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Arithmetic Review: Factors, Products, and Exponents
This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. This chapter contains many examples of arithmetic techniques that are used directly or indirectly in algebra. Since the chapter is intended as a review, the problem-solving techniques are presented without being developed. Therefore, no work space is provided, nor does the chapter contain all of the pedagogical features of the text. As a review, this chapter can be assigned at the discretion of the instructor and can also be a valuable reference tool for the student.

Overview

- Factors
- Exponential Notation

Factors

Let’s begin our review of arithmetic by recalling the meaning of multiplication for whole numbers (the counting numbers and zero).

Multiplication
Multiplication is a description of repeated addition.

In the addition

$$7 + 7 + 7 + 7$$

the number 7 is repeated as an **addend*** 4 **times.** Therefore, we say we have **four times seven** and describe it by writing

$$4 \cdot 7$$

The raised dot between the numbers 4 and 7 indicates multiplication. The dot directs us to multiply the two numbers that it separates. In algebra, the dot is preferred over the symbol $\times$ to denote multiplication because the letter x is often used to represent a number. Thus,

$$4 \cdot 7 = 7 + 7 + 7 + 7$$

Factors and Products
In a multiplication, the numbers being multiplied are called **factors.** The result of a multiplication is called the **product.** For example, in the multiplication

$$4 \cdot 7 = 28$$

the numbers 4 and 7 are factors, and the number 28 is the product. We say that 4 and 7 are factors of 28. (They are not the only factors of 28. Can you think of others?)

Now we know that

$$(\text{factor}) \cdot (\text{factor}) \quad = \quad \text{product}$$

This indicates that a first number is a factor of a second number if the first number divides into the second number with no remainder. For example, since

$$4 \cdot 7 = 28$$

both 4 and 7 are factors of 28 since both 4 and 7 divide into 28 with no remainder.

Exponential Notation

Quite often, a particular number will be repeated as a factor in a multiplication. For example, in the multiplication

$$7 \cdot 7 \cdot 7 \cdot 7$$

the number 7 is repeated as a factor 4 times. We describe this by writing 7^4. Thus,

$7 \cdot 7 \cdot 7 \cdot 7 = 7^4$

The repeated factor is the lower number (the base), and the number recording how many times the factor is repeated is the higher number (the superscript). The superscript number is called an **exponent.**

Exponent
An **exponent** is a number that records how many times the number to which it is attached occurs as a factor in a multiplication.

Sample Set A

For Examples 1, 2, and 3, express each product using exponents.

Example:
$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3.$ Since 3 occurs as a factor 6 times,

$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^6$

Example:
$8 \cdot 8.$ Since 8 occurs as a factor 2 times,

$8 \cdot 8 = 8^2$

Example:
$5 \cdot 5 \cdot 5 \cdot 9 \cdot 9.$ Since 5 occurs as a factor 3 times, we have 5^3. Since 9 occurs as a factor 2 times, we have 9^2. We should see the following replacements.

$$\underbrace{5 \cdot 5 \cdot 5}_{5^3} \cdot \underbrace{9 \cdot 9}_{9^2}$$

Then we have

$$5 \cdot 5 \cdot 5 \cdot 9 \cdot 9 = 5^3 \cdot 9^2$$

Example:
Expand 3^5. The base is 3 so it is the repeated factor. The exponent is 5 and it records the number of times the base 3 is repeated. Thus, 3 is to be repeated as a factor 5 times.

$$3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$$

Example:
Expand $6^2 \cdot 10^4$. The notation $6^2 \cdot 10^4$ records the following two facts: 6 is to be repeated as a factor 2 times and 10 is to be repeated as a factor 4 times. Thus,

$$6^2 \cdot 10^4 = 6 \cdot 6 \cdot 10 \cdot 10 \cdot 10 \cdot 10$$

Exercises

For the following problems, express each product using exponents.

Exercise:

Problem:$8 \cdot 8 \cdot 8$

Solution:

8^3

Exercise:

Problem:$12 \cdot 12 \cdot 12 \cdot 12 \cdot 12$

Exercise:

Problem:$5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$

Solution:

5^7

Exercise:

Problem:$1 \cdot 1$

Exercise:

Problem:$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 4 \cdot 4$

Solution:

$3^5 \cdot 4^2$

Exercise:

Problem:$8 \cdot 8 \cdot 8 \cdot 15 \cdot 15 \cdot 15 \cdot 15$

Exercise:

Problem:$2 \cdot 2 \cdot 2 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 \cdot 9$

Solution:

$2^3 \cdot 9^8$

Exercise:

Problem:$3 \cdot 3 \cdot 10 \cdot 10 \cdot 10$

Exercise:

Problem:

Suppose that the letters x and y are each used to represent numbers. Use exponents to express the following product.

$x \cdot x \cdot x \cdot y \cdot y$

Solution:

$x^3 \cdot y^2$

Exercise:

Problem:

Suppose that the letters x and y are each used to represent numbers. Use exponents to express the following product.

$x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y$

For the following problems, expand each product (do not compute the actual value).

Exercise:

Problem:3^4

Solution:

$3 \cdot 3 \cdot 3 \cdot 3$

Exercise:

Problem:4^3

Exercise:

Problem:2^5

Solution:

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

Exercise:

Problem:9^6

Exercise:

Problem:$5^3 \cdot 6^2$

Solution:

$5 \cdot 5 \cdot 5 \cdot 6 \cdot 6$

Exercise:

Problem:$2^7 \cdot 3^4$

Exercise:

Problem:$x^4 \cdot y^4$

Solution:

$x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y$

Exercise:

Problem:$x^6 \cdot y^2$

For the following problems, specify all the whole number factors of each number. For example, the complete set of whole number factors of 6 is 1, 2, 3, 6.

Exercise:

Problem:20

Solution:

$1, 2, 4, 5, 10, 20$

Exercise:

Problem:14

Exercise:

Problem:12

Solution:

$1, 2, 3, 4, 6, 12$

Exercise:

Problem:30

Exercise:

Problem:21

Solution:

$1, 3, 7, 21$

Exercise:

Problem:45

Exercise:

Problem:11

Solution:

1, 11

Exercise:

Problem:17

Exercise:

Problem:19

Solution:

1, 19

Exercise:

Problem:2

Arithmetic Review: Prime Factorization
This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. This chapter contains many examples of arithmetic techniques that are used directly or indirectly in algebra. Since the chapter is intended as a review, the problem-solving techniques are presented without being developed. Therefore, no work space is provided, nor does the chapter contain all of the pedagogical features of the text. As a review, this chapter can be assigned at the discretion of the instructor and can also be a valuable reference tool for the student.

Overview

- Prime And Composite Numbers
- The Fundamental Principle Of Arithmetic
- The Prime Factorization Of A Whole Number

Prime And Composite Numbers

Notice that the only factors of 7 are 1 and 7 itself, and that the only factors of 23 are 1 and 23 itself.

Prime Number

A whole number greater than 1 whose only whole number factors are itself and 1 is called a **prime number.**

The first seven prime numbers are

2, 3, 5, 7, 11, 13, and 17

The number 1 is not considered to be a prime number, and the number 2 is the first and only even prime number.
Many numbers have factors other than themselves and 1. For example, the factors of 28 are 1, 2, 4, 7, 14, and 28 (since each of these whole numbers and only these whole numbers divide into 28 without a remainder).

Composite Numbers

A whole number that is composed of factors other than itself and 1 is called a **composite number.** Composite numbers are not prime numbers.

Some composite numbers are 4, 6, 8, 10, 12, and 15.

The Fundamental Principle Of Arithmetic

Prime numbers are very important in the study of mathematics. We will use them soon in our study of fractions. We will now, however, be introduced to an important mathematical principle.

The Fundamental Principle of Arithmetic
Except for the order of the factors, every whole number, other than 1, can be factored in one and only one way as a product of prime numbers.

Prime Factorization
When a number is factored so that all its factors are prime numbers, the factorization is called the **prime factorization** of the number.

Sample Set A

Example:
Find the prime factorization of 10.

$10 = 2 \cdot 5$

Both 2 and 5 are prime numbers. Thus, $2 \cdot 5$ is the prime factorization of 10.

Example:
Find the prime factorization of 60.

$$\begin{aligned}60 &= 2 \cdot 30 && 30\text{ is not prime. } 30 = 2 \cdot 15\\ &= 2 \cdot 2 \cdot 15 && 15\text{ is not prime. } 15 = 3 \cdot 5\\ &= 2 \cdot 2 \cdot 3 \cdot 5 && \text{We'll use exponents. } 2 \cdot 2 = 2^2\\ &= 2^2 \cdot 3 \cdot 5\end{aligned}$$

The numbers 2, 3, and 5 are all primes. Thus, $2^2 \cdot 3 \cdot 5$ is the prime factorization of 60.

Example:
Find the prime factorization of 11.

11 is a prime number. Prime factorization applies only to composite numbers.

The Prime Factorization Of A Whole Number

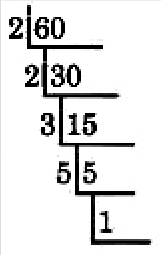
The following method provides a way of finding the prime factorization of a whole number. The examples that follow will use the method and make it more clear.

1. Divide the number repeatedly by the smallest prime number that will divide into the number without a remainder.
2. When the prime number used in step 1 no longer divides into the given number without a remainder, repeat the process with the next largest prime number.
3. Continue this process until the quotient is 1.
4. The prime factorization of the given number is the product of all these prime divisors.

Sample Set B

Example:
Find the prime factorization of 60.

Since 60 is an even number, it is divisible by 2. We will repeatedly divide by 2 until we no longer can (when we start getting a remainder). We shall divide in the following way.



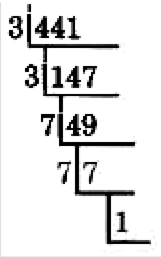
30 is divisible by 2 again.
15 is not divisible by 2, but is divisible by 3, the next largest prime.
5 is not divisible by 3, but is divisible by 5, the next largest prime.
The quotient is 1 so we stop the division process.

The prime factorization of 60 is the product of all these divisors.

$$\begin{aligned} 60 &= 2 \cdot 2 \cdot 3 \cdot 5 \qquad \text{We will use exponents when possible.} \\ 60 &= 2^2 \cdot 3 \cdot 5 \end{aligned}$$

Example:
Find the prime factorization of 441.

Since 441 is an odd number, it is not divisible by 2. We’ll try 3, the next largest prime.



$$\begin{array}{l} 147 \text{ is divisible by } 3. \\ 49 \text{ is not divisible by } 3 \text{ nor by } 5, \text{ but by } 7. \\ 7 \text{ is divisible by } 7. \\ \text{The quotient is } 1 \text{ so we stop the division process.} \end{array}$$

The prime factorization of 441 is the product of all the divisors.

$$\begin{array}{lcll} 441 & = & 3 \cdot 3 \cdot 7 \cdot 7 & \text{We will use exponents when possible.} \\ 441 & = & 3^2 \cdot 7^2 & \end{array}$$

Exercises

For the following problems, determine which whole numbers are prime and which are composite.

Exercise:

Problem:23

Solution:

prime

Exercise:

Problem:25

Exercise:

Problem:27

Solution:

composite

Exercise:

Problem:2

Exercise:

Problem:3

Solution:

prime

Exercise:

Problem:5

Exercise:

Problem:7

Solution:

prime

Exercise:

Problem:9

Exercise:

Problem:11

Solution:

prime

Exercise:

Problem:34

Exercise:

Problem:55

Solution:

composite

Exercise:

Problem:63

Exercise:

Problem:1044

Solution:

composite

Exercise:

Problem:339

Exercise:

Problem:209

Solution:

composite

For the following problems, find the prime factorization of each whole number. Use exponents on repeated factors.

Exercise:

Problem:26

Exercise:

Problem:38

Solution:

$2 \cdot 19$

Exercise:

Problem:54

Exercise:

Problem:62

Solution:

$2 \cdot 31$

Exercise:

Problem:56

Exercise:

Problem:176

Solution:

$2^4 \cdot 11$

Exercise:

Problem:480

Exercise:

Problem:819

Solution:

$3^2 \cdot 7 \cdot 13$

Exercise:

Problem:2025

Exercise:

Problem:148,225

Solution:

$5^2 \cdot 7^2 \cdot 11^2$

Arithmetic Review: The Least Common Multiple
This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr.
This chapter contains many examples of arithmetic techniques that are used directly or indirectly in algebra. Since the chapter is intended as a review, the problem-solving techniques are presented without being developed. Therefore, no work space is provided, nor does the chapter contain all of the pedagogical features of the text. As a review, this chapter can be assigned at the discretion of the instructor and can also be a valuable reference tool for the student.

Overview

- Multiples
- Common Multiples
- The Least Common Multiple (LCM)
- Finding The Least Common Multiple

Multiples

Multiples
When a whole number is multiplied by other whole numbers, with the exception of Multiples zero, the resulting products are called **multiples** of the given whole number.

Multiples of 2		**Multiples of 3**		**Multiples of 8**		**Multiples of 10**
$2 \cdot 1 = 2$		$3 \cdot 1 = 3$		$8 \cdot 1 = 8$		$10 \cdot 1 = 10$
$2 \cdot 2 = 4$		$3 \cdot 2 = 6$		$8 \cdot 2 = 16$		$10 \cdot 2 = 20$
$2 \cdot 3 = 6$		$3 \cdot 3 = 9$		$8 \cdot 3 = 24$		$10 \cdot 3 = 30$

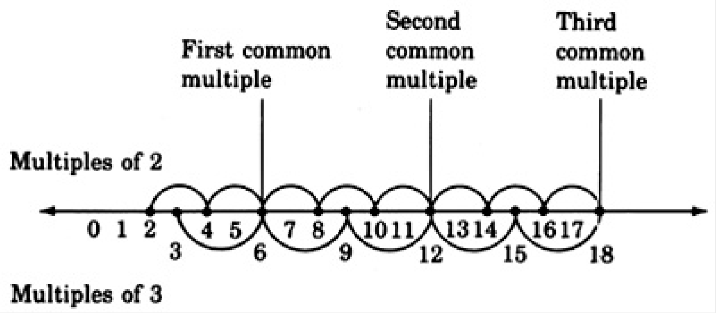
$2 \cdot 4 = 8$		$3 \cdot 4 = 12$		$8 \cdot 4 = 32$		$10 \cdot 4 = 40$
$2 \cdot 5 = 10$		$3 \cdot 5 = 15$		$8 \cdot 5 = 40$		$10 \cdot 5 = 50$
…		…		…		…

Common Multiples

There will be times when we are given two or more whole numbers and we will need to know if there are any multiples that are common to each of them. If there are, we will need to know what they are. For example, some of the multiples that are common to 2 and 3 are 6, 12, and 18.

Sample Set A

Example:
We can visualize common multiples using the number line.



Notice that the common multiples can be divided by both whole numbers.

The Least Common Multiple (LCM)

Notice that in our number line visualization of common multiples (above) the first common multiple is also the smallest, or **least common multiple,** abbreviated by **LCM.**

Least Common Multiple
The **least common multiple, LCM,** of two or more whole numbers is the smallest whole number that each of the given numbers will divide into without a remainder.

Finding The Least Common Multiple

Finding the LCM
To find the LCM of two or more numbers,

1. Write the prime factorization of each number, using exponents on repeated factors.
2. Write each base that appears in each of the prime factorizations.
3. To each base, attach the largest exponent that appears on it in the prime factorizations.
4. The LCM is the product of the numbers found in step 3.

Sample Set B

Find the LCM of the following number.

Example:
9 and 12

1. $$\begin{aligned} 9 &= 3 \cdot 3 = 3^2 \\ 12 &= 2 \cdot 6 = 2 \cdot 2 \cdot 3 = 2^2 \cdot 3 \end{aligned}$$
2. The bases that appear in the prime factorizations are 2 and 3.
3. The largest exponents appearing on 2 and 3 in the prime factorizations are, respectively, 2 and 2 (or 2^2 from 12, and 3^2 from 9).
4. The LCM is the product of these numbers.

   $$\text{LCM} = 2^2 \cdot 3^2 = 4 \cdot 9 = 36$$

Thus, 36 is the smallest number that both 9 and 12 divide into without remainders.

Example:
90 and 630

1. $$\begin{aligned} 90 &= 2 \cdot 45 &&= 2 \cdot 3 \cdot 15 &&= 2 \cdot 3 \cdot 3 \cdot 5 &&= 2 \cdot 3^2 \cdot 5 \\ 630 &= 2 \cdot 315 &&= 2 \cdot 3 \cdot 105 &&= 2 \cdot 3 \cdot 3 \cdot 35 &&= 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \\ & && && &&= 2 \cdot 3^2 \cdot 5 \cdot 7 \end{aligned}$$
2. The bases that appear in the prime factorizations are 2, 3, 5, and 7.
3. The largest exponents that appear on 2, 3, 5, and 7 are, respectively, 1, 2, 1, and 1.
   $$\begin{aligned} &2^1 \quad \text{from either 90 or 630} \\ &3^2 \quad \text{from either 90 or 630} \\ &5^1 \quad \text{from either 90 or 630} \\ &7^1 \quad \text{from 630} \end{aligned}$$
4. The LCM is the product of these numbers.
   $$\text{LCM } = 2 \cdot 3^2 \cdot 5 \cdot 7 = 2 \cdot 9 \cdot 5 \cdot 7 = 630$$

Thus, 630 is the smallest number that both 90 and 630 divide into with no remainders.

Example:
33, 110, and 484

1. $$\begin{aligned} 33 &= 3 \cdot 11 \\ 110 &= 2 \cdot 55 &&= 2 \cdot 5 \cdot 11 \\ 484 &= 2 \cdot 242 &&= 2 \cdot 2 \cdot 121 = 2 \cdot 2 \cdot 11 \cdot 11 = 2^2 \cdot 11^2 \end{aligned}$$
2. The bases that appear in the prime factorizations are 2, 3, 5, and 11.
3. The largest exponents that appear on 2, 3, 5, and 11 are, respectively, 2, 1, 1, and 2.

$$\begin{array}{ll} 2^2 & \text{from } 484 \\ 3^1 & \text{from } 33 \\ 5^1 & \text{from } 110 \\ 11^2 & \text{from } 484 \end{array}$$

4. The LCM is the product of these numbers.

$$\begin{aligned} \text{LCM} &= 2^2 \cdot 3 \cdot 5 \cdot 11^2 \\ &= 4 \cdot 3 \cdot 5 \cdot 121 \\ &= 7260 \end{aligned}$$

Thus, 7260 is the smallest number that 33, 110, and 484 divide into without remainders.

Exercises

For the following problems, find the least common multiple of given numbers.

Exercise:

Problem:8, 12

Solution:

$2^3 \cdot 3$

Exercise:

Problem:8, 10

Exercise:

Problem:6, 12

Solution:

$2^2 \cdot 3$

Exercise:

Problem: 9, 18

Exercise:

Problem: 5, 6

Solution:

$2 \cdot 3 \cdot 5$

Exercise:

Problem: 7, 9

Exercise:

Problem: 28, 36

Solution:

$2^2 \cdot 3^2 \cdot 7$

Exercise:

Problem: 24, 36

Exercise:

Problem: 28, 42

Solution:

$2^2 \cdot 3 \cdot 7$

Exercise:

Problem: 20, 24

Exercise:

Problem: 25, 30

Solution:

$2 \cdot 3 \cdot 5^2$

Exercise:

Problem:24, 54

Exercise:

Problem:16, 24

Solution:

$2^4 \cdot 3$

Exercise:

Problem:36, 48

Exercise:

Problem:15, 21

Solution:

$3 \cdot 5 \cdot 7$

Exercise:

Problem:7, 11, 33

Exercise:

Problem:8, 10, 15

Solution:

$2^3 \cdot 3 \cdot 5$

Exercise:

Problem:4, 5, 21

Exercise:

Problem:45, 63, 98

Solution:

$2 \cdot 3^2 \cdot 5 \cdot 7^2$

Exercise:

Problem:15, 25, 40

Exercise:

Problem:12, 16, 20

Solution:

$2^4 \cdot 3 \cdot 5$

Exercise:

Problem:12, 16, 24

Exercise:

Problem:12, 16, 24, 36

Solution:

$2^4 \cdot 3^2$

Exercise:

Problem:6, 9, 12, 18

Exercise:

Problem:8, 14, 28, 32

Solution:

$2^5 \cdot 7$

Arithmetic Review: Equivalent Fractions
This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. This chapter contains many examples of arithmetic techniques that are used directly or indirectly in algebra. Since the chapter is intended as a review, the problem-solving techniques are presented without being developed. Therefore, no work space is provided, nor does the chapter contain all of the pedagogical features of the text. As a review, this chapter can be assigned at the discretion of the instructor and can also be a valuable reference tool for the student.

Overview

- Equivalent Fractions
- Reducing Fractions To Lowest Terms
- Raising Fractions To Higher Terms

Equivalent Fractions

Equivalent Fractions

Fractions that have the same value are called **equivalent fractions.**

For example, $\frac{2}{3}$ and $\frac{4}{6}$ represent the same part of a whole quantity and are therefore equivalent. Several more collections of equivalent fractions are listed below.

Example:
$\frac{15}{25}, \frac{12}{20}, \frac{3}{5}$

Example:
$\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}$

Example:
$\frac{7}{6}, \frac{14}{12}, \frac{21}{18}, \frac{28}{24}, \frac{35}{30}$

Reducing Fractions To Lowest Terms

Reduced to Lowest Terms

It is often useful to convert one fraction to an equivalent fraction that has reduced values in the numerator and denominator. When a fraction is converted to an equivalent fraction that has the smallest numerator and denominator in the collection of equivalent fractions, it is said to be **reduced to lowest terms.** The conversion process is called **reducing a fraction.**

We can reduce a fraction to lowest terms by

1. Expressing the numerator and denominator as a product of prime numbers. (Find the prime factorization of the numerator and denominator. See Section ([link]) for this technique.)
2. Divide the numerator and denominator by all common factors. (This technique is commonly called "cancelling.")

Sample Set A

Reduce each fraction to lowest terms.

Example:

$$\begin{aligned} \frac{6}{18} &= \frac{2\cdot 3}{2\cdot 3\cdot 3} \\ &= \frac{\cancel{2}\cdot\cancel{3}}{\cancel{2}\cdot\cancel{3}\cdot 3} && \text{2 and 3 are common factors.} \\ &= \frac{1}{3} \end{aligned}$$

Example:

$$\begin{aligned} \frac{16}{20} &= \frac{2\cdot 2\cdot 2\cdot 2}{2\cdot 2\cdot 5} \\ &= \frac{\cancel{2}\cdot\cancel{2}\cdot 2\cdot 2}{\cancel{2}\cdot\cancel{2}\cdot 5} && \text{2 is the only common factor.} \\ &= \frac{4}{5} \end{aligned}$$

Example:

$$\begin{aligned} \frac{56}{70} &= \frac{2\cdot 4\cdot 7}{2\cdot 5\cdot 7} \\ &= \frac{\cancel{2}\cdot 4\cdot\cancel{7}}{\cancel{2}\cdot 5\cdot\cancel{7}} && \text{2 and 7 are common factors.} \\ &= \frac{4}{5} \end{aligned}$$

Example:

$$\begin{aligned} \frac{8}{15} &= \frac{2\cdot 2\cdot 2}{3\cdot 5} && \text{There are no common factors.} \\ &\text{Thus, } \frac{8}{15} \text{ is reduced to lowest terms.} \end{aligned}$$

Raising a Fraction to Higher Terms

Equally important as reducing fractions is **raising fractions to higher terms.** Raising a fraction to higher terms is the process of constructing an equivalent fraction that has higher values in the numerator and denominator. The higher, equivalent fraction is constructed by multiplying the original fraction by 1.

Notice that $\frac{3}{5}$ and $\frac{9}{15}$ are equivalent, that is $\frac{3}{5}=\frac{9}{15}$. Also,

$$\frac{3}{5}\cdot\underbrace{1=\frac{3}{5}\cdot\frac{3}{3}}_{1=\frac{3}{3}}=\frac{3\cdot 3}{5\cdot 3}=\frac{9}{15}$$

This observation helps us suggest the following method for raising a fraction to higher terms.

Raising a Fraction to Higher Terms
A fraction can be raised to higher terms by multiplying both the numerator and denominator by the same nonzero number.

For example, $\frac{3}{4}$ can be raised to $\frac{24}{32}$ by multiplying both the numerator and denominator by 8, that is, multiplying by 1 in the form $\frac{8}{8}$.

$$\frac{3}{4}=\frac{3\cdot 8}{4\cdot 8}=\frac{24}{32}$$

How did we know to choose 8 as the proper factor? Since we wish to convert 4 to 32 by multiplying it by some number, we know that 4 must be a factor of 32. This means that 4 divides into 32. In fact, $32\div 4=8$. We divided the original denominator into the new, specified denominator to obtain the proper factor for the multiplication.

Sample Set B

Determine the missing numerator or denominator.

Example:
$\frac{3}{7}=\frac{?}{35}.\quad$ Divide the original denominator, 7, into the new denominator, 35. $35\div 7=5$.
Multiply the original numerator by 5.
$\frac{3}{7}=\frac{3\cdot 5}{7\cdot 5}=\frac{15}{35}$

Example:
$\frac{5}{6} = \frac{45}{?}.\quad \text{Divide the original numerator, } 5, \text{ into the new numerator, } 45.\ 45 \div 5 = 9.$
$\text{Multiply the original denominator by } 9.$
$\frac{5}{6} = \frac{5 \cdot 9}{6 \cdot 9} = \frac{45}{54}$

Exercises

For the following problems, reduce, if possible, each fraction lowest terms.

Exercise:

Problem: $\frac{6}{8}$

Solution:

$\frac{3}{4}$

Exercise:

Problem: $\frac{5}{10}$

Exercise:

Problem: $\frac{6}{14}$

Solution:

$\frac{3}{7}$

Exercise:

Problem: $\frac{4}{14}$

Exercise:

Problem: $\frac{18}{12}$

Solution:

$\frac{3}{2}$

Exercise:

Problem: $\frac{20}{8}$

Exercise:

Problem: $\frac{10}{6}$

Solution:

$\frac{5}{3}$

Exercise:

Problem: $\frac{14}{4}$

Exercise:

Problem: $\frac{10}{12}$

Solution:

$\frac{5}{6}$

Exercise:

Problem: $\frac{32}{28}$

Exercise:

Problem: $\frac{36}{10}$

Solution:

$\frac{18}{5}$

Exercise:

Problem: $\frac{26}{60}$

Exercise:

Problem: $\frac{12}{18}$

Solution:

$\frac{2}{3}$

Exercise:

Problem: $\frac{18}{27}$

Exercise:

Problem: $\frac{18}{24}$

Solution:

$\frac{3}{4}$

Exercise:

Problem: $\frac{32}{40}$

Exercise:

Problem: $\frac{11}{22}$

Solution:

$\frac{1}{2}$

Exercise:

Problem: $\frac{17}{51}$

Exercise:

Problem: $\frac{27}{81}$

Solution:

$\frac{1}{3}$

Exercise:

Problem: $\frac{16}{42}$

Exercise:

Problem: $\frac{39}{13}$

Solution:

3

Exercise:

Problem: $\frac{44}{11}$

Exercise:

Problem: $\frac{121}{132}$

Solution:

$\frac{11}{12}$

Exercise:

Problem: $\frac{30}{105}$

Exercise:

Problem: $\frac{108}{76}$

Solution:

$\frac{27}{19}$

For the following problems, determine the missing numerator or denominator.

Exercise:

Problem: $\frac{1}{3} = \frac{?}{12}$

Exercise:

Problem: $\frac{1}{5} = \frac{?}{30}$

Solution:

6

Exercise:

Problem: $\frac{3}{3} = \frac{?}{9}$

Exercise:

Problem: $\frac{3}{4} = \frac{?}{16}$

Solution:

12

Exercise:

Problem: $\frac{5}{6} = \frac{?}{18}$

Exercise:

Problem: $\frac{4}{5} = \frac{?}{25}$

Solution:

20

Exercise:

Problem: $\frac{1}{2} = \frac{4}{?}$

Exercise:

Problem: $\frac{9}{25} = \frac{27}{?}$

Solution:

75

Exercise:

Problem: $\frac{3}{2} = \frac{18}{?}$

Exercise:

Problem: $\frac{5}{3} = \frac{80}{?}$

Solution:

48

Arithmetic Review: Operations with Fractions
This module is from Elementary Algebra by Denny Burzynski and Wade Ellis, Jr. This chapter contains many examples of arithmetic techniques that are used directly or indirectly in algebra. Since the chapter is intended as a review, the problem-solving techniques are presented without being developed. Therefore, no work space is provided, nor does the chapter contain all of the pedagogical features of the text. As a review, this chapter can be assigned at the discretion of the instructor and can also be a valuable reference tool for the student.

Overview

- Multiplication of Fractions
- Division of Fractions
- Addition and Subtraction of Fractions

Multiplication of Fractions

Multiplication of Fractions
To multiply two fractions, multiply the numerators together and multiply the denominators together. Reduce to lowest terms if possible.

Example:
For example, multiply $\frac{3}{4} \cdot \frac{1}{6}$.

$$\begin{array}{rlll} \frac{3}{4} \cdot \frac{1}{6} & = & \frac{3 \cdot 1}{4 \cdot 6} & \\ & = & \frac{3}{24} & \text{Now reduce.} \\ & = & \frac{3 \cdot 1}{2 \cdot 2 \cdot 2 \cdot 3} & \\ & = & \frac{\cancel{3} \cdot 1}{2 \cdot 2 \cdot 2 \cdot \cancel{3}} & 3 \text{ is the only common factor.} \\ & = & \frac{1}{8} & \end{array}$$

Notice that we since had to reduce, we nearly started over again with the original two fractions. If we factor first, then cancel, then multiply, we will save time and energy and still obtain the correct product.

Sample Set A

Perform the following multiplications.

Example:

$$\begin{aligned}\frac{1}{4}\cdot\frac{8}{9} &= \frac{1}{2\cdot 2}\cdot\frac{2\cdot 2\cdot 2}{3\cdot 3}\\ &= \frac{1}{\cancel{2}\cdot\cancel{2}}\cdot\frac{\cancel{2}\cdot\cancel{2}\cdot 2}{3\cdot 3} \qquad \text{2 is a common factor.}\\ &= \frac{1}{1}\cdot\frac{2}{3\cdot 3}\\ &= \frac{1\cdot 2}{1\cdot 3\cdot 3}\\ &= \frac{2}{9}\end{aligned}$$

Example:

$$\begin{aligned}\frac{3}{4}\cdot\frac{8}{9}\cdot\frac{5}{12} &= \frac{3}{2\cdot 2}\cdot\frac{2\cdot 2\cdot 2}{3\cdot 3}\cdot\frac{5}{2\cdot 2\cdot 3}\\ &= \frac{\cancel{3}}{\cancel{2}\cdot\cancel{2}}\cdot\frac{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}}{\cancel{3}\cdot 3}\cdot\frac{5}{\cancel{2}\cdot 2\cdot 3} \qquad \text{2 and 3 are common factors.}\\ &= \frac{1\cdot 1\cdot 5}{3\cdot 2\cdot 3}\\ &= \frac{5}{18}\end{aligned}$$

Division of Fractions

Reciprocals
Two numbers whose product is 1 are **reciprocals** of each other. For example, since $\frac{4}{5}\cdot\frac{5}{4}=1$, $\frac{4}{5}$ and $\frac{5}{4}$ are reciprocals of each other. Some other pairs of reciprocals are listed below.

$$\frac{2}{7},\frac{7}{2} \qquad \frac{3}{4},\frac{4}{3} \qquad \frac{6}{1},\frac{1}{6}$$

Reciprocals are used in division of fractions.

Division of Fractions
To divide a first fraction by a second fraction, multiply the first fraction by the reciprocal of the second fraction. Reduce if possible.

This method is sometimes called the “invert and multiply” method.

Sample Set B

Perform the following divisions.

Example:

$\frac{1}{3} \div \frac{3}{4}$. The divisor is $\frac{3}{4}$. Its reciprocal is $\frac{4}{3}$.

$$\begin{aligned} \frac{1}{3} \div \frac{3}{4} &= \frac{1}{3} \cdot \frac{4}{3} \\ &= \frac{1 \cdot 4}{3 \cdot 3} \\ &= \frac{4}{9} \end{aligned}$$

Example:

$\frac{3}{8} \div \frac{5}{4}$. The divisor is $\frac{5}{4}$. Its reciprocal is $\frac{4}{5}$.

$$\begin{aligned} \frac{3}{8} \div \frac{5}{4} &= \frac{3}{8} \cdot \frac{4}{5} \\ &= \frac{3}{2 \cdot 2 \cdot 2} \cdot \frac{2 \cdot 2}{5} \\ &= \frac{3}{\cancel{2} \cdot \cancel{2} \cdot 2} \cdot \frac{\cancel{2} \cdot \cancel{2}}{5} && \text{2 is a common factor.} \\ &= \frac{3 \cdot 1}{2 \cdot 5} \\ &= \frac{3}{10} \end{aligned}$$

Example:

$\frac{5}{6} \div \frac{5}{12}$. The divisor is $\frac{5}{12}$. Its reciprocal is $\frac{12}{5}$.

$$\begin{aligned} \frac{5}{6} \div \frac{5}{12} &= \frac{5}{6} \cdot \frac{12}{5} \\ &= \frac{5}{2 \cdot 3} \cdot \frac{2 \cdot 2 \cdot 3}{5} \\ &= \frac{\cancel{5}}{\cancel{2} \cdot \cancel{3}} \cdot \frac{\cancel{2} \cdot 2 \cdot \cancel{3}}{\cancel{5}} \\ &= \frac{1 \cdot 2}{1} \\ &= 2 \end{aligned}$$

Addition and Subtraction of Fractions

Fractions with Like Denominators

To add (or subtract) two or more fractions that have the same denominators, add (or subtract) the numerators and place the resulting sum over the common denominator. Reduce if possible.

CAUTION

Add or subtract only the numerators. Do **not** add or subtract the denominators!

Sample Set C

Find the following sums.

Example:

$\frac{3}{7} + \frac{2}{7}$. The denominators are the same. Add the numerators and place the sum over 7.

$\frac{3}{7} + \frac{2}{7} = \frac{3+2}{7} = \frac{5}{7}$

Example:

$\frac{7}{9} - \frac{4}{9}$. The denominators are the same. Subtract 4 from 7 and place the difference over 9.

$\frac{7}{9} - \frac{4}{9} = \frac{7-4}{9} = \frac{3}{9} = \frac{1}{3}$

Fractions can only be added or subtracted conveniently if they have like denominators.

Fractions with Unlike Denominators
To add or subtract fractions having unlike denominators, convert each fraction to an equivalent fraction having as the denominator the least common multiple of the original denominators.

The least common multiple of the original denominators is commonly referred to as the **least common denominator** (LCD). See Section ([link]) for the technique of finding the least common multiple of several numbers.

Sample Set D

Find each sum or difference.

Example:

$\frac{1}{6} + \frac{3}{4}$. The denominators are not alike. Find the LCD of 6 and 4.

$\begin{cases} 6 = 2 \cdot 3 \\ 4 = 2^2 \end{cases}$ The LCD is $2^2 \cdot 3 = 4 \cdot 3 = 12$.

Convert each of the original fractions to equivalent fractions having the common denominator 12.

$\frac{1}{6} = \frac{1 \cdot 2}{6 \cdot 2} = \frac{2}{12} \qquad \frac{3}{4} = \frac{3 \cdot 3}{4 \cdot 3} = \frac{9}{12}$

Now we can proceed with the addition.

$$\begin{aligned} \frac{1}{6} + \frac{3}{4} &= \frac{2}{12} + \frac{9}{12} \\ &= \frac{2+9}{12} \\ &= \frac{11}{12} \end{aligned}$$

Example:

$\frac{5}{9} - \frac{5}{12}.$ The denominators are not alike. Find the LCD of 9 and 12.

$\begin{cases} 9 = 3^2 \\ 12 = 2^2 \cdot 3 \end{cases}$ The LCD is $2^2 \cdot 3^2 = 4 \cdot 9 = 36.$

Convert each of the original fractions to equivalent fractions having the common denominator 36.

$\frac{5}{9} = \frac{5 \cdot 4}{9 \cdot 4} = \frac{20}{36} \qquad \frac{5}{12} = \frac{5 \cdot 3}{12 \cdot 3} = \frac{15}{36}$

Now we can proceed with the subtraction.

$$\begin{aligned} \frac{5}{9} - \frac{5}{12} &= \frac{20}{36} - \frac{15}{36} \\ &= \frac{20-15}{36} \\ &= \frac{5}{36} \end{aligned}$$

Exercises

For the following problems, perform each indicated operation.

Exercise:

Problem: $\frac{1}{3} \cdot \frac{4}{3}$

Solution:

$\frac{4}{9}$

Exercise:

Problem: $\frac{1}{3} \cdot \frac{2}{3}$

Exercise:

Problem: $\frac{2}{5} \cdot \frac{5}{6}$

Solution:

$\frac{1}{3}$

Exercise:

Problem: $\frac{5}{6} \cdot \frac{14}{15}$

Exercise:

Problem: $\frac{9}{16} \cdot \frac{20}{27}$

Solution:

$\frac{5}{12}$

Exercise:

Problem: $\frac{35}{36} \cdot \frac{48}{55}$

Exercise:

Problem: $\frac{21}{25} \cdot \frac{15}{14}$

Solution:

$\frac{9}{10}$

Exercise:

Problem: $\frac{76}{99} \cdot \frac{66}{38}$

Exercise:

Problem: $\frac{3}{7} \cdot \frac{14}{18} \cdot \frac{6}{2}$

Solution:

1

Exercise:

Problem: $\frac{14}{15} \cdot \frac{21}{28} \cdot \frac{45}{7}$

Exercise:

Problem: $\frac{5}{9} \div \frac{5}{6}$

Solution:

$\frac{2}{3}$

Exercise:

Problem: $\frac{9}{16} \div \frac{15}{8}$

Exercise:

Problem: $\frac{4}{9} \div \frac{6}{15}$

Solution:

$\frac{10}{9}$

Exercise:

Problem: $\frac{25}{49} \div \frac{4}{9}$

Exercise:

Problem: $\frac{15}{4} \div \frac{27}{8}$

Solution:

$\frac{10}{9}$

Exercise:

Problem: $\frac{24}{75} \div \frac{8}{15}$

Exercise:

Problem: $\frac{57}{8} \div \frac{7}{8}$

Solution:

$\frac{57}{7}$

Exercise:

Problem: $\frac{7}{10} \div \frac{10}{7}$

Exercise:

Problem: $\frac{3}{8} + \frac{2}{8}$

Solution:

$\frac{5}{8}$

Exercise:

Problem: $\frac{3}{11} + \frac{4}{11}$

Exercise:

Problem: $\frac{5}{12} + \frac{7}{12}$

Solution:

1

Exercise:

Problem: $\frac{11}{16} - \frac{2}{16}$

Exercise:

Problem: $\frac{15}{23} - \frac{2}{23}$

Solution:

$\frac{13}{23}$

Exercise:

Problem: $\frac{3}{11} + \frac{1}{11} + \frac{5}{11}$

Exercise:

Problem: $\frac{16}{20} + \frac{1}{20} + \frac{2}{20}$

Solution:

$\frac{19}{20}$

Exercise:

Problem: $\frac{3}{8} + \frac{2}{8} - \frac{1}{8}$

Exercise:

Problem: $\frac{11}{16} + \frac{9}{16} - \frac{5}{16}$

Solution:

$\frac{15}{16}$

Exercise:

Problem: $\frac{1}{2} + \frac{1}{6}$

Exercise:

Problem: $\frac{1}{8} + \frac{1}{2}$

Solution:

$\frac{5}{8}$

Exercise:

Problem: $\frac{3}{4} + \frac{1}{3}$

Exercise:

Problem: $\frac{5}{8} + \frac{2}{3}$

Solution:

$\frac{31}{24}$

Exercise:

Problem: $\frac{6}{7} - \frac{1}{4}$

Exercise:

Problem: $\frac{8}{15} - \frac{3}{10}$

Solution:

$\frac{5}{6}$

Exercise:

Problem: $\frac{1}{15} + \frac{5}{12}$

Exercise:

Problem: $\frac{25}{36} - \frac{7}{10}$

Solution:

$\frac{-1}{180}$

Exercise:

Problem: $\frac{9}{28} - \frac{4}{45}$

Exercise:

Problem: $\frac{8}{15} - \frac{3}{10}$

Solution:

$\frac{7}{30}$

Exercise:

Problem: $\frac{1}{16} + \frac{3}{4} - \frac{3}{8}$

Exercise:

Problem: $\frac{8}{3} - \frac{1}{4} + \frac{7}{36}$

Solution:

$\frac{47}{18}$

Exercise:

Problem: $\frac{3}{4} - \frac{3}{22} + \frac{5}{24}$

Adding and Subtracting Fractions with Like and Unlike Denominators, and LCD

This module discusses how to add and subtract fractions with like denominators and how to find the least common denominator to allow addition and subtraction of fractions with unlike denominators.

Adding Fractions with Like Denominators

To add two or more fractions that have the same denominators, add the numerators and place the resulting sum over the **common denominator**. Reduce, if necessary.

Example:
Find the following sums.

$\frac{3}{7}+\frac{2}{7}$

The denominators are the same.
Add the numerators and place the sum over the common denominator, 7.

$\frac{3}{7}+\frac{2}{7}=\frac{3+2}{7}=\frac{5}{7}$

When necessary, reduce the result.

Example:

$\frac{1}{8}+\frac{3}{8}=\frac{1+3}{8}=\frac{4}{8}=\frac{1}{2}$

Note: We do not add denominators.

Example:

To see what happens if we **mistakenly** add the denominators as well as the numerators, let’s add
$\frac{1}{2}$ and $\frac{1}{2}$.
Adding the numerators and **mistakenly** adding the denominators produces:
$\frac{1}{2} + \frac{1}{2} = \frac{1+1}{2+2} = \frac{2}{4} = \frac{1}{2}$
This means that $\frac{1}{2} + \frac{1}{2}$ is the same as $\frac{1}{2}$, which is preposterous! **We do not add denominators**.

Adding Fractions with Like Denominators - Exercises

Find the following sums.

Exercise:

Problem: $\frac{3}{8} + \frac{3}{8}$

Solution:

$\frac{6}{8} = \frac{3}{4}$

Exercise:

Problem: $\frac{7}{11} + \frac{4}{11}$

Solution:

$\frac{11}{11} = 1$

Exercise:

Problem: $\frac{15}{20} + \frac{1}{20} + \frac{2}{20}$

Solution:

$\frac{18}{20} = \frac{9}{10}$

Subtracting Fractions with Like Denominators

To subtract two or more fractions that have the same denominators, subtract the numerators and place the resulting difference over the **common denominator**. Reduce, if necessary.

Example:
Find the following differences.
$\frac{3}{5}$ - $\frac{1}{5}$
The denominators are the same.
Subtract the numerators and place the difference over the common denominator, 5.
$\frac{3}{5}$ - $\frac{1}{5}$ $= \frac{3-1}{5} = \frac{2}{5}$

When necessary, reduce the result.

Example:
$\frac{8}{6}$ - $\frac{2}{6}$ $= \frac{6}{6}$ = 1

Note: We do not subtract denominators.

Example:
To see what happens if we **mistakenly** subtract the denominators as well as the numerators, let’s subtract

$\frac{7}{15}$ - $\frac{4}{15}$.

Subtracting the numerators and **mistakenly** subtracting the denominators produces:

$\frac{7}{15}$ - $\frac{4}{15}$ $= \frac{7-4}{15-15} = \frac{3}{0}$

We end up dividing by zero, which is undefined. **We do not subtract denominators.**

Subtracting Fractions with Like Denominators - Exercises

Find the following differences.

Exercise:

Problem: $\frac{5}{12}$ - $\frac{1}{12}$

Solution:

$\frac{4}{12} = \frac{1}{3}$

Exercise:

Problem: $\frac{3}{16}$ - $\frac{3}{16}$

Solution:

Result is 0

Exercise:

Problem: $\frac{16}{5}$ - $\frac{1}{5}$ - $\frac{2}{5}$

Solution:

Result is $\frac{13}{5}$

Adding and Subtracting Fractions with Unlike Denominators

Basic Rule: Fractions can only be added or subtracted conveniently if they have like denominators.

To see why this rule makes sense, let’s consider the problem of adding a quarter and a dime.

A quarter is $\frac{1}{4}$ of a dollar.

A dime is $\frac{1}{10}$ of a dollar.

We know that 1 quarter + 1 dime = 35 cents. How do we get to this answer by adding $\frac{1}{4}$ and $\frac{1}{10}$?

We convert them to quantities of the same denomination.

A quarter is equivalent to 25 cents, or $\frac{25}{100}$.

A dime is equivalent to 10 cents, or $\frac{10}{100}$.

By converting them to quantities of the same denomination, we can add them easily:

$\frac{25}{100} + \frac{10}{100} = \frac{35}{100}$.

Same denomination $\rightarrow$ same denominator

If the denominators are not the same, make them the same by building up the fractions so that they both have a common denominator. A common denominator can always be found by multiplying all the denominators, but it is not necessarily the Least Common Denominator.

Least Common Denominator (LCD)

The LCD is the smallest number that is evenly divisible by all the denominators.

It is the **least common multiple** of the denominators.

The LCD is the product of all the **prime factors** of all the denominators, each factor taken the greatest number of times that it appears in any single denominator.

Finding the LCD

Example:
Find the sum of these unlike fractions.
$\frac{1}{12} + \frac{4}{15}$
Factor the denominators:
12 = 2 × 2 × 3
15 = 3 × 5
What is the greatest number of times the prime factor 2 appear in any single denominator? **Answer: 2times.** That is the number of times the prime factor 2 will appear as a factor in the LCD.
What is the greatest number of times the prime factor 3 appear in any single denominator? **Answer: 1 time.** That is the number of times the prime factor 3 will appear as a factor in the LCD.
What is the greatest number of times the prime factor 5 appear in any single denominator? **Answer: 1 time.** That is the number of times the prime factor 5 will appear as a factor in the LCD.
So we assemble the LCD by multiplying each prime factor by the number of times it appears in a single denominator, or:
2 × 2 × 3 × 5 = 60
60 is the Least Common Denominator (the Least Common Multiple of 12 and 15).

Building up the Fractions

To create fractions with like denominators, we now multiply the numerators by whatever factors are missing when we compare the original denominator to the new LCD.

Example:
In the fraction $\frac{1}{12}$, we multiply the denominator 12 by 5 to get the LCD of 60. Therefore we multiply the numerator 1 by the same factor (5).
$\frac{1}{12} \times \frac{5}{5} = \frac{5}{60}$
Similarly,
$\frac{4}{15} \times \frac{4}{4} = \frac{16}{60}$

Adding the Built Up Fractions

Example:
We can now add the two fractions because they have like denominators:
$\frac{5}{60} + \frac{16}{60} = \frac{21}{60}$
Reduce the result: $\frac{21}{60} = \frac{7}{20}$

Adding and Subtracting Fractions with Unlike Denominators - Exercises

Find the following sums and differences.

Exercise:

Problem: $\frac{1}{6} + \frac{3}{4}$

Solution:

Result is $\frac{11}{12}$

Exercise:

Problem: $\frac{5}{9} - \frac{5}{12}$

Solution:

Result is $\frac{5}{36}$

Exercise:

Problem: $\frac{15}{16} + \frac{1}{2} - \frac{3}{4}$

Solution:

Result is $\frac{35}{16}$

Module Review Exercises

Exercise:

Problem: $\frac{9}{15} + \frac{4}{15}$

Solution:

Result is $\frac{13}{15}$

Exercise:

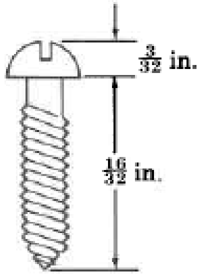
Problem: $\frac{7}{10} - \frac{3}{10} + \frac{11}{10}$

Solution:

Result is $\frac{15}{10}$ (reduce to 1 $\frac{1}{2}$)

Exercise:

Problem: Find the total length of the screw in this diagram:



Solution:

Total length is $\frac{19}{32}$ in.

Exercise:

Problem: $\frac{5}{2} + \frac{16}{2} - \frac{3}{2}$

Solution:

Result is $\frac{18}{2}$ (reduce to 9)

Exercise:

Problem: $\frac{3}{4} + \frac{1}{3}$

Solution:

Result is $\frac{13}{12}$

Exercise:

Problem:

Two months ago, a woman paid off $\frac{3}{24}$ of a loan. One month ago, she paid off $\frac{4}{24}$ of the loan. This month she will pay off $\frac{5}{24}$ of the total loan. At the end of this month, how much of her total loan will she have paid off?

Solution:

She will have paid off $\frac{12}{24}$, or $\frac{1}{2}$ of the total loan.

Exercise:

Problem: $\frac{8}{3} - \frac{1}{4} + \frac{7}{36}$

Solution:

Result is $\frac{94}{36}$ (reduce to $\frac{47}{18}$)

Signed Numbers: Absolute Value
This module is from Fundamentals of Mathematics by Denny Burzynski and Wade Ellis, Jr. This module discusses absolute value. By the end of the module students should understand the geometric and algebraic definitions of absolute value.

Section Overview

- Geometric Definition of Absolute Value
- Algebraic Definition of Absolute Value

Geometric Definition of Absolute Value

Absolute Value-Geometric Approach

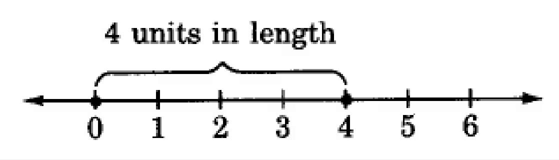
Geometric definition of absolute value:
The **absolute value** of a number a, denoted $\mid a \mid$, is the distance from a to 0 on the number line.

Absolute value answers the question of "how far," and not "which way." The phrase "how far" implies "length" and *length is always a nonnegative quantity*. Thus, the absolute value of a number is a nonnegative number.

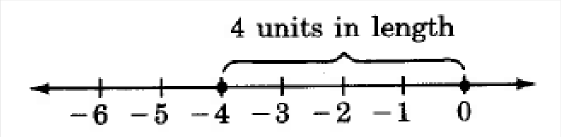
Sample Set A

Determine each value.

Example:
$\mid 4 \mid = 4$



Example:
$|-4|=4$



Example:
$|0|=0$

Example:
$-|5|=-5$. The quantity on the left side of the equal sign is read as "negative the absolute value of 5." The absolute value of 5 is 5. Hence, negative the absolute value of 5 is -5.

Example:
$-|-3|=-3$. The quantity on the left side of the equal sign is read as "negative the absolute value of -3." The absolute value of -3 is 3. Hence, negative the absolute value of -3 is $-(3)=-3$.

Practice Set A

By reasoning geometrically, determine each absolute value.
Exercise:

Problem: $|7|$

Solution:

7

Exercise:

Problem: $|\ -3\ |$

Solution:

3

Exercise:

Problem: $|\ 12\ |$

Solution:

12

Exercise:

Problem: $|\ 0\ |$

Solution:

0

Exercise:

Problem: $-\ |\ 9\ |$

Solution:

-9

Exercise:

Problem: $-\ |\ -6\ |$

Solution:

-6

Algebraic Definition of Absolute Value

From the problems in [link], we can suggest the following algebraic definition of absolute value. Note that the definition has two parts.

Absolute Value—Algebraic Approach
Algebraic definition of absolute value
The absolute value of a number a is

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } < 0 \end{cases}$$

The algebraic definition takes into account the fact that the number a could be either positive or zero $(a \geq 0)$ or negative $(a < 0)$.

1. If the number a is positive or zero $(a \geq 0)$, the upper part of the definition applies. The upper part of the definition tells us that if the number enclosed in the absolute value bars is a nonnegative number, the absolute value of the number is the number itself.
2. The lower part of the definition tells us that if the number enclosed within the absolute value bars is a negative number, the absolute value of the number is the opposite of the number. The opposite of a negative number is a positive number.

Note: The definition says that the vertical absolute value lines may be eliminated only if we know whether the number inside is positive or negative.

Sample Set B

Use the algebraic definition of absolute value to find the following values.

Example:
$\mid 8 \mid$. The number enclosed within the absolute value bars is a nonnegative number, so the upper part of the definition applies. This part says that the absolute value of 8 is 8 itself.
$\mid 8 \mid = 8$

Example:
$\mid -3 \mid$. The number enclosed within absolute value bars is a negative number, so the lower part of the definition applies. This part says that the absolute value of -3 is the opposite of -3, which is $-(-3)$. By the definition of absolute value and the double-negative property,
$\mid -3 \mid = -(-3) = 3$

Practice Set B

Use the algebraic definition of absolute value to find the following values.
Exercise:

Problem: $\mid 7 \mid$

Solution:

7

Exercise:

Problem: $\mid 9 \mid$

Solution:

9

Exercise:

Problem: $|-12|$

Solution:

12

Exercise:

Problem: $|-5|$

Solution:

5

Exercise:

Problem: $-|8|$

Solution:

-8

Exercise:

Problem: $-|1|$

Solution:

-1

Exercise:

Problem: $-|-52|$

Solution:

-52

Exercise:

Problem: $- \mid -31 \mid$

Solution:

-31

Exercises

Determine each of the values.

Exercise:

Problem: $\mid 5 \mid$

Solution:

5

Exercise:

Problem: $\mid 3 \mid$

Exercise:

Problem: $\mid 6 \mid$

Solution:

6

Exercise:

Problem: $\mid -9 \mid$

Exercise:

Problem: $|-1|$

Solution:

1

Exercise:

Problem: $|-4|$

Exercise:

Problem: $-|3|$

Solution:

-3

Exercise:

Problem: $-|7|$

Exercise:

Problem: $-|-14|$

Solution:

-14

Exercise:

Problem: $|0|$

Exercise:

Problem: $|-26|$

Solution:

26

Exercise:

Problem: $-\mid -26\mid$

Exercise:

Problem: $-(-\mid 4\mid)$

Solution:

4

Exercise:

Problem: $-(-\mid 2\mid)$

Exercise:

Problem: $-(-\mid -6\mid)$

Solution:

6

Exercise:

Problem: $-(-\mid -42\mid)$

Exercise:

Problem: $\mid 5\mid - \mid -2\mid$

Solution:

3

Exercise:

Problem: $\mid -2 \mid^3$

Exercise:

Problem: $\mid -(2 \cdot 3) \mid$

Solution:

6

Exercise:

Problem: $\mid -2 \mid - \mid -9 \mid$

Exercise:

Problem: $(\mid -6 \mid + \mid 4 \mid)^2$

Solution:

100

Exercise:

Problem: $(\mid -1 \mid - \mid 1 \mid)^3$

Exercise:

Problem: $(\mid 4 \mid + \mid -6 \mid)^2 - (\mid -2 \mid)^3$

Solution:

92

Exercise:

Problem: $-\left[\left|-10\right|-6\right]^2$

Exercise:

Problem: $-\left\{-\left[-\left|-4\right|+\left|-3\right|\right]^3\right\}^2$

Solution:

-1

Exercise:

Problem:

A Mission Control Officer at Cape Canaveral makes the statement “lift-off, *T* minus 50 seconds.” How long is it before lift-off?

Exercise:

Problem:

Due to a slowdown in the industry, a Silicon Valley computer company finds itself in debt \$2,400,000. Use absolute value notation to describe this company’s debt.

Solution:

$-\$\left|-2{,}400{,}000\right|$

Exercise:

Problem:

A particular machine is set correctly if upon action its meter reads 0. One particular machine has a meter reading of -1.6 upon action. How far is this machine off its correct setting?

Exercises for Review

Exercise:

Problem: ([link]) Find the sum: $\frac{9}{70}+\frac{5}{21}+\frac{8}{15}$.

Solution:

$\frac{9}{10}$

Exercise:

Problem: ([link]) Find the value of $\frac{\frac{3}{10}+\frac{4}{12}}{\frac{19}{20}}$.

Exercise:

Problem: ([link]) Convert $3.2\frac{3}{5}$ to a fraction.

Solution:

$3\frac{13}{50}\text{or}\frac{163}{50}$

Exercise:

Problem:

([link]) The ratio of acid to water in a solution is $\frac{3}{8}$. How many mL of acid are there in a solution that contain 112 mL of water?

Exercise:

Problem: ([link]) Find the value of $-6-(-8)$.

Solution:

2

Exponent Concepts -- Introduction

An exponent means repeated multiplication. For instance, 10^6 means $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$, or 1,000,000.

You’ve probably noticed that there is a logical progression of operations. When you **add the same number repeatedly**, that’s multiplication. When you **multiply the same number repeatedly**, that’s an exponent.

However, there is one vital difference: addition and multiplication **commute**, but exponentiation does **not commute**. This is a fancy way of saying that order matters. $2 + 3 = 3 + 2$; and $2 \cdot 3 = 3 \cdot 2$, but 2^3 is **not** the same as 3^2.

Exponent Concepts -- Laws of Exponents

This module covers the laws the govern the manipulation of exponents in algebra.

The following are generally referred to as the “laws” or “rules” of exponents.

Equation:

$$x^a x^b = x^{a+b}$$

Equation:

$$\frac{x^a}{x^b} = x^{a-b} \qquad \text{or} \qquad \frac{1}{x^{b-a}}$$

Equation:

$$\left(x^a\right)^b = x^{\text{ab}}$$

As with any formula, the most important thing is to be able to **use** them—that is, to understand what they mean. But it is also important to know **where these formulae come from**. And finally, in this case, the three should be memorized.

So...what do they mean? They are, of course, algebraic generalizations—statements that are true for any x, a, and b values. For instance, the first rule tells us that:

Equation:

$$7^{12} \cdot 7^4 = 7^{16}$$

which you can confirm on your calculator. Similarly, the third rule promises us that

Equation:

$$\left(7^{12}\right)^4 = 7^{48}$$

These rules can be used to combine and simplify expressions.

Example:

Simplifying with the Rules of Exponents	
$\frac{\left(x^3\right)^4 \cdot x^5}{x^9 \cdot x^{11}}$	
$= \frac{x^{12} \cdot x^5}{x^9 \cdot x^{11}}$	**Third rule**: $\left(x^a\right)^b = x^{\text{ab}}$, **where**$a = 3$**and**$b = 4$
$= \frac{x^{17}}{x^{20}}$	**First rule**: $x^a x^b = x^{a+b}$, **done on both the top and bottom**
$= \frac{1}{x^3}$	**Second rule**: $\frac{x^a}{x^b} = \frac{1}{x^{b-a}}$, **where we choose this form to avoid a negative** exponent

Why do these rules work? It's very easy to see, based on what an exponent is.

Why does the first rule work?

Why does the first rule work?	
19^3	19^4
$= (19 \cdot 19 \cdot 19)$	$(19 \cdot 19 \cdot 19 \cdot 19)$
$= 19^7$	

You see what happened? 19^3 means three 19s multiplied; 19^4 means four 19s multiplied. Multiply them together, and you get **seven** 19s multiplied.

Why does the second rule work?	
First form	Second form
$\frac{19^8}{19^5}$	$\frac{19^5}{19^8}$
$= \frac{19\cdot19\cdot19\cdot19\cdot19\cdot19\cdot19\cdot19}{19\cdot19\cdot19\cdot19\cdot19}$	$= \frac{19\cdot19\cdot19\cdot19\cdot19}{19\cdot19\cdot19\cdot19\cdot19\cdot19\cdot19\cdot19}$
$= \frac{\cancel{19}\cdot\cancel{19}\cdot\cancel{19}\cdot\cancel{19}\cdot\cancel{19}\cdot 19\cdot 19\cdot 19}{\cancel{19}\cdot\cancel{19}\cdot\cancel{19}\cdot\cancel{19}\cdot\cancel{19}}$	$= \frac{\cancel{19}\cdot\cancel{19}\cdot\cancel{19}\cdot\cancel{19}\cdot\cancel{19}}{\cancel{19}\cdot\cancel{19}\cdot\cancel{19}\cdot\cancel{19}\cdot\cancel{19}\cdot 19\cdot 19\cdot 19}$
$= 19^3$	$= \frac{1}{19^3}$

In this case, the key is **fraction cancellations**. When the top is multiplied by 19 and the bottom is multiplied by 19, canceling these 19s has the effect of dividing **the top and bottom** by 19. When you divide the top and bottom of a fraction by the same number, the fraction is unchanged.

You can also think of this rule as the inevitable consequence of the **first** rule. If $19^3 \cdot 19^5 = 19^8$, then $\frac{19^8}{19^5}$ (which asks the question “ 19^5 times what equals 19^8?”) must be 19^3.

Why does the third rule work?
$\left(19^3\right)^4$
$= (19 \cdot 19 \cdot 19) \cdot (19 \cdot 19 \cdot 19) \cdot (19 \cdot 19 \cdot 19) \cdot (19 \cdot 19 \cdot 19)$
$= 19^{12}$

What does it mean to raise something to the fourth power? It means to multiply it by itself, four times. In this case, what we are multiplying by itself four times is 19^3, or $(19 \cdot 19 \cdot 19)$. Three 19s multiplied **four times** makes twelve 19s multiplied.

Exponent Concepts -- Zero, negative numbers, and fractions as exponents

This module covers how zero, negative numbers, and fractions can be used as exponents.

The definition of exponents given at the beginning of this section— 10^6means $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$—does **not** enable us to answer questions such as:

Equation:

$$4^0 = ?$$

Equation:

$$5^{-4} = ?$$

Equation:

$$9^{\frac{1}{2}} = ?$$

You can't "multiply 9 by itself half a time" or "multiply 5 by itself -4 times." In general, the original definition only applies if the exponent is a **positive integer**.

It's very important to understand this point. The question is **not** "What answer does our original definition give in these cases?" The original definition does **not** give any answer in these cases. If we want to include these numbers, we need a whole new definition of what an exponent is.

In principle, many such definitions are possible. We **could** define 5^{-4} as $5/5/5/5$: in other words, divide four times instead of multiplying four times. Or we **could** define 5^{-4}as 5^{-4}: take 5 to the fourth power, and then multiply it by -1. Or we **could** define 5^{-4} as $-(5)^4$: take -5 to the fourth power (which gives a different answer from the previous definition). It seems at first that we are at liberty to choose any definition we want.

Given that degree of freedom, you may be very surprised at the definitions that are actually used: they seem far more arbitrary and complicated than some others you could come up with.

Zero exponents	**Negative exponents**	**Fractional exponents (numerator = 1)**	**Fractional exponents (numerator ≠ 1)**
Always 1	Go in the denominator	Act as roots	The **numerator** is an exponent The **denominator** is a root
$7^0 = 1$ $9^0 = 1$ $x^0 = 1$	$7^{-3} = \frac{1}{7^3} = \frac{1}{343}$ $x^{-5} = \frac{1}{x^5}$ $\frac{1}{5^{-3}} = 5^3$	$9^{\frac{1}{2}} = \sqrt{9} = 3$ $2^{\frac{1}{2}} = \sqrt{2}$ $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$ $x^{\frac{1}{4}} = \sqrt[4]{x}$	$8^{\frac{2}{3}} = \sqrt[3]{8^2}$ **or** $\left(\sqrt[3]{8}\right)^2$ Order doesn't matter! $\sqrt[3]{8^2} = \sqrt[3]{64} = 4$ **or** $\left(\sqrt[3]{8}\right)^2 = 2^2 = 4$ $8^{\frac{3}{2}} = \sqrt{8^3}$ **or** $\left(\sqrt{8}\right)^3$

Definitions: When the exponent is not a positive integer

Note that you can **combine** these definitions. For instance, $8^{-\frac{2}{3}}$ is a negative, fractional exponent. The negative exponent means, as always, "put me in the denominator." So we can write:
Equation:

$$8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{8^2}} = \frac{1}{4}$$

OK, so why define exponents that way?

These are obviously not chosen to be the **simplest** possible definitions. But they are chosen to be **consistent** with the behavior of positive-integer exponents.

One way to see that consistency is to consider the following progression:
Equation:

$$19^4 = 19 \cdot 19 \cdot 19 \cdot 19$$

Equation:

$$19^3 = 19 \cdot 19 \cdot 19$$

Equation:

$$19^2 = 19 \cdot 19$$

Equation:

$$19^1 = 19$$

What happens each time we decrease the exponent by 1? Your first response might be "we have one less 19." But what is really happening, mathematically, to the numbers on the right? The answer is that, with each step, they are **dividing by** 19. If you take $19 \cdot 19 \cdot 19 \cdot 19$, and **divide it by** 19, you get $19 \cdot 19 \cdot 19$. Divide that by 19 again, and you get $19 \cdot 19$...and so on. From this we can formulate the following principle for the powers of 19:

Whenever you **subtract 1** from the exponent, you **divide the answer by** 19.

As I said earlier, we want the behavior of our new exponents to be consistent with the behavior of the **old** (positive-integer) exponents. So we can continue this progression as follows:
Equation:

$$19^0 = \frac{19}{19} = 1$$

Equation:

$$19^{-1} = \frac{1}{19}$$

Equation:

$$19^{-2} = \frac{\frac{1}{19}}{19} = \frac{1}{19^2}$$

...and so on. We can arrive at our definitions $\text{anything}^0 = 1$ and **negative exponents go in the denominator** by simply requiring this progression to be consistent.

More rigorously, we can find all our exponent definitions by using the **laws of exponents**. For instance, what is 4^0? We can approach this question indirectly by asking: what is $\frac{4^2}{4^2}$?

- The second law of exponents tells us that $\frac{4^2}{4^2} = 4^{2-2}$, which is of course 4^0.
- But of course, $\frac{4^2}{4^2}$ is just $\frac{16}{16}$, or 1.
- Since $\frac{4^2}{4^2}$ is both 4^0**and** 1, 4^0 and 1 must be the same thing!

The proofs given below all follow this pattern. They use the laws of exponents to rewrite expressions such as $\frac{4^2}{4^2}$, and go on to show how zero, negative, and fractional exponents must be defined. We started with the definition of an exponent for a positive integer, $10^6 = 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$. From there, we developed the laws of exponents. Now we find that, if we want those same laws to apply to other kinds of exponents, there is only one correct way to define those other kinds of exponents.

Zero exponents	**Negative exponents**	**Fractional exponents (numerator = 1)**	**Fractional exponents (numerator ≠ 1)**
Always 1	Go in the denominator	Act as roots	The **numerator** is an exponent The **denominator** is a root
$\frac{4^2}{4^2} = 4^{2-2} = 4^0$ **but** $\frac{4^2}{4^2} = \frac{16}{16} = 1$ **so** 4^0 must be 1!	$\frac{10^1}{10^3} = 10^{1-3} = 10^{-2}$ **but** $\frac{10^1}{10^3} = \frac{10}{10 \cdot 10 \cdot 10} = \frac{1}{10 \cdot 10}$ **so** 10^{-2} must be $\frac{1}{10^2}$	$\left(9^{\frac{1}{2}}\right)^2 = 9^{\frac{1}{2} \cdot 2} = 9^1 = 9$ **So** what is $9^{\frac{1}{2}}$? Well, when you **square** it, you get 9. So it must be $\sqrt{9}$, or 3!	$8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{8}\right)^2$ **or** $8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}} = \sqrt[3]{8^2}$

Proofs: When the exponent is not a positive integer

You may want to experiment with making these proofs more general and more rigorous by using letters instead of numbers. For instance, in the third case, we could write:

Equation:

$$(x^{\frac{1}{a}})^a = x^{\left(\frac{1}{a}\right)(a)} = x^1$$

Equation:

$$(x^{\frac{1}{a}})^a = x$$

Equation:

$$\sqrt[a]{(x)^a} = \sqrt[a]{x}$$

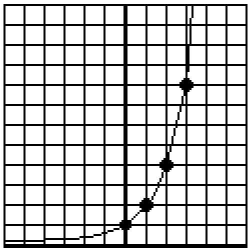
Equation:

$$x^{\frac{1}{a}} = \sqrt[a]{x}$$

Exponent Concepts -- Exponential Curves

This module discusses the graphing of exponential curves.

By plotting points, you can discover that the graph of $y=2^x$ looks like this:

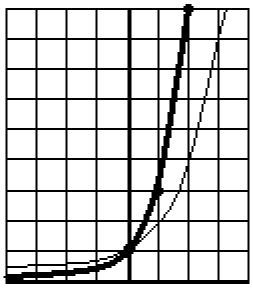


$y=2^x$

A few points to notice about this graph.

- It goes through the point $(0{,}1)$ because $2^0=1$.
- It never dips below the x-axis. The **domain** is unlimited, but the **range** is y>0. (*Think about our definitions of exponents: whether x is positive or negative, integer or fraction, 2^x is **always** positive.)
- Every time you move one unit to the right, the graph height doubles. For instance, 2^5 is twice 2^4, because it multiplies by **one more** 2. So as you move to the right, the y-values start looking like 8, 16, 32, 64, 128, and so on, going up more and more sharply.
- Conversely, every time you move one unit to the left, the graph height drops in half. So as you move to the left, the y-values start looking like $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, and so on, falling closer and closer to 0.

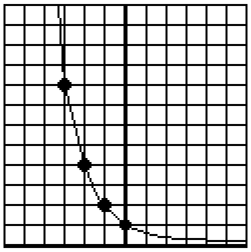
What would the graph of $y=3^x$ look like? Of course, it would also go through $(0{,}1)$ because $3^0=1$. With each step to the right, it would **triple**; with each step to the left, it would drop in a **third**. So the overall shape would look similar, but the rise (on the right) and the drop (on the left) would be faster.



$y = 2^x$ in thin line; $y = 3^x$ in thick line; They cross at $(0,1)$

As you might guess, graphs such as 5^x and 10^x all have this same characteristic shape. In fact, any graph a^x where $a > 1$ will look basically the same: starting at $(0,1)$ it will rise more and more sharply on the right, and drop toward zero on the left. This type of graph models **exponential growth**—functions that keep multiplying by the same number. A common example, which you work through in the text, is compound interest from a bank.

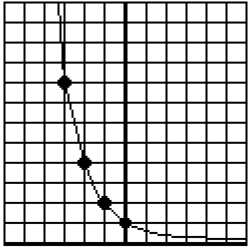
The opposite graph is $\left(\frac{1}{2}\right)^x$.



$y = \left(\frac{1}{2}\right)^x$

Each time you move to the right on this graph, it multiplies by $\frac{1}{2}$: in other words, it **divides** by 2, heading closer to zero the further you go. This kind of equation is used to model functions that keep **dividing** by the same number; for instance, radioactive decay. You will also be working through examples like this one.

Of course, all the **permutations** from the first chapter on “functions” apply to these graphs just as they apply to any graph. A particularly interesting example is 2^{-x}. Remember that when you replace x with $-x$, $f(3)$ becomes the old $f(-3)$ and vice-versa; in other words, the graph flips around the y-axis. If you take the graph of 2^x and permute it in this way, you get a familiar shape:



$y = 2^{-x}$

Yes, it’s $\left(\frac{1}{2}\right)^x$ in a new disguise!

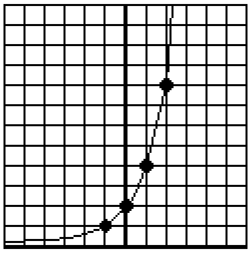
Why did it happen that way? Consider that $\left(\frac{1}{2}\right)^x = \frac{1^x}{2^x}$. But 1^x is just 1 (in other words, 1 to the **anything** is 1), so $\left(\frac{1}{2}\right)^x = \frac{1}{2^x}$. But negative exponents go in the denominator: $\frac{1}{2^x}$ is the same thing as 2^{-x}! So we arrive at: $\left(\frac{1}{2}\right)^x = 2^{-x}$. The two functions are the same, so their graphs are of course the same.

Another fun pair of permutations is:

$y = 2 \cdot 2^x$ **Looks just like** $y = 2^x$ **but vertically stretched: all y-values double**

$y = 2^{x+1}$ **Looks just like** $y = 2^x$ **but horizontally shifted: moves 1 to the left**

If you permute 2^x in these two ways, you will find that they create the same graph.



$y = 2 \cdot 2^x$
aka
$y = 2^{x+1}$

Once again, this is predictable from the rules of exponents:
$2 \cdot 2^x = 2^1 \cdot 2^x = 2^{x+1}$

Using exponential functions to model behavior

In the first chapter, we talked about **linear functions** as functions that **add the same amount every time**. For instance, $y = 3\text{x} + 4$ models a function that starts at 4; every time you increase x by 1, you add 3 to y.

Exponential functions are conceptually very analogous: they **multiply by the same amount every time**. For instance, $y = 4 \times 3^x$ models a function that starts at 4; every time you increase x by 1, you **multiply**y by 3.

Linear functions can go down, as well as up, by having **negative slopes**: $y = -3\text{x} + 4$ starts at 4 and **subtracts** 3 every time. Exponential functions can go down, as well as up, by having **fractional bases**: $y = 4 \times (\frac{1}{3})^x$ starts at 4 and **divides** by 3 every time.

Exponential functions often defy intuition, because they grow much faster than people expect.

Modeling exponential functions

Your father’s house was worth $100,000 when he bought it in 1981. Assuming that it increases in value by 8% every year, what was the house worth in the year 2001? (*Before you work through the math, you may want to make an intuitive guess as to what you think the house is worth. Then, after we crunch the numbers, you can check to see how close you got.)

Often, the best way to approach this kind of problem is to begin by making a chart, to get a sense of the growth pattern.

Year	Increase in Value	Value
1981	N/A	100,000
1982	8% of 100,000 = 8,000	108,000
1983	8% of 108,000 = 8,640	116,640
1984	8% of 116,640 = 9,331	125,971

Before you go farther, make sure you understand where the numbers on that chart come from. It’s OK to use a calculator. But if you blindly follow the numbers without understanding the calculations, the whole rest of this section will be lost on you.

In order to find the pattern, look at the “Value” column and ask: what is happening to these numbers every time? Of course, we are adding 8% each time, but what does that really mean? With a little thought—or by looking at the numbers—you should be able to convince yourself that the numbers are **multiplying by** 1.08 **each time**. That's why this is an exponential function: the value of the house multiplies by 1.08 every year.

So let's make that chart again, in light of this new insight. Note that I can now skip the middle column and go straight to the answer we want. More importantly, note that I am not going to use my calculator this time—I don't want to multiply all those 1.08s, I just want to note each time that the answer is 1.08 **times the previous answer**.

Year	**House Value**
1981	100,000
1982	$100{,}000 \times 1.08$
1983	$100{,}000 \times 1.08^2$
1984	$100{,}000 \times 1.08^3$
1985	$100{,}000 \times 1.08^4$
y	$100{,}000 \times 1.08^{\text{something}}$

If you are not clear where those numbers came from, think again about the conclusion we reached earlier: each year, the value **multiplies by** 1.08. So if the house is worth $100{,}000 \times 1.08^2$in 1983, then its value in 1984 is $\left(100{,}000 \times 1.08^2\right) \times 1.08$, which is $100{,}000 \times 1.08^3$.

Once we write it this way, the pattern is clear. I have expressed that pattern by adding the last row, the value of the house in any year y. And what is the mystery exponent? We see that the exponent is 1 in 1982, 2 in 1983, 3 in 1984, and so on. In the year y, the exponent is $y - 1981$.

So we have our house value function:
Equation:

$$v(y) = 100{,}000 \times 1.08^{y-1981}$$

That is the pattern we needed in order to answer the question. So in the year 2001, the value of the house is $100{,}000 \times 1.08^{20}$. Bringing the calculator back, we find that the value of the house is now $466,095 and change.

Wow! The house is over four times its original value! That’s what I mean about exponential functions growing faster than you expect: they start out slow, but given time, they explode. This is also a practical life lesson about the importance of saving money early in life—a lesson that many people don’t realize until it’s too late.

Function Concepts -- Introduction

This module provides an introduction to the concept of functions.

The unit on functions is the most important in the Algebra II course, because it provides a crucial transition point. Roughly speaking…

- Before Algebra I, math is about **numbers**.
- Starting in Algebra I, and continuing into Algebra II, math is about **variables**.
- Beginning with Algebra II, and continuing into Calculus, math is about **functions**.

Each step builds on the previous step. Each step expands the ability of mathematics to model behavior and solve problems. And, perhaps most crucially, each step can be frightening to a student. It can be very intimidating for a beginning Algebra student to see an entire page of mathematics that is covered with letters, with almost no numbers to be found!

Unfortunately, many students end up with a very vague idea of what variables are ("That's when you use letters in math") and an even more vague understanding of functions ("Those things that look like $f(x)$ or something"). If you leave yourself with this kind of vague understanding of the core concepts, the lessons will make less and less sense as you go on: you will be left with the feeling that "I just can't do this stuff" without realizing that the problem was all the way back in the idea of a variable or function.

The good news is, variables and functions both have very specific meanings that are not difficult to understand.

Function Concepts -- What is a Variable?

This module defines what a variable is and what it does.

A variable is a letter that stands for a number you don’t know, or a number that can change.

A few examples:

Example:
Good Examples of Variable Definitions

- “Let p be the number of people in a classroom.”
- “Let A be John’s age, measured in years.”
- “Let h be the number of hours that Susan has been working.”

In each case, the letter stands for a very specific number. However, we use a letter instead of a number because we don’t know the specific number. In the first example above, different classrooms will have different numbers of people (so p can be different numbers in different classes); in the second example, John’s age is a specific and well-defined number, but we don’t know what it is (at least not yet); and in the third example, h will actually change its value every hour. In all three cases, we have a good reason for using a letter: it represents a number, but we cannot use a specific number such as “–3” or “ $4\frac{1}{2}$ ”.

Example:
Bad Examples of Variable Definitions

- “Let n be the nickels.”
- “Let M be the number of minutes in an hour.”

The first error is by far the most common. Remember that a variable always stands for a number. “The nickels” are not a number. Better definitions would be: “Let n be the number of nickels” or “Let n be the total value of the nickels, measured in cents” or “Let n be the total mass of the nickels, measured in grams.”

The second example is better, because “number of minutes in an hour” is a number. But there is no reason to call it “The Mysterious Mr. M” because we already know what it is. Why use a letter when you just mean “60”?

Bad variable definitions are one of the most common reasons that students get stuck on word problems—or get the wrong answer. The first type of error illustrated above leads to variable confusion: n will end up being used for “number of nickels” in one equation and “total value of the nickels” in another, and you end up with the wrong answer. The second type of error is more harmless—it won’t lead to wrong answers—but it won’t help either. It usually indicates that the student is asking the wrong question (“What can I assign a variable to?”) instead of the right question (“What numbers do I need to know?”)

Variables aren’t all called x. Get used to it.

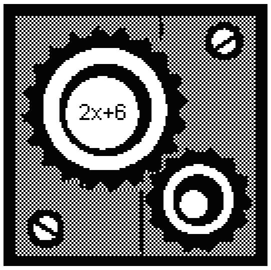
Many students expect all variables to be named x, with possibly an occasional guest appearance by y. In fact, variables can be named with practically any letter. Uppercase letters, lowercase letters, and even Greek letters are commonly used for variable names. Hence, a problem might start with “Let H be the home team’s score and V be the visiting team’s score.”

If you attempt to call both of these variables x, it just won’t work. You could in principle call one of them x and the other y, but that would make it more difficult to remember which variable goes with which team. It is important to become comfortable using a wide range of letters. (I do, however, recommend avoiding the letter o whenever possible, since it looks like the number 0.)

Function Concepts -- What is a Function?

A function is neither a number nor a variable: it is a **process for turning one number into another**. For instance, “Double and then add 6” is a function. If you put a 4 into that function, it comes out with a 14. If you put a $\frac{1}{2}$ into that function, it comes out with a 7.

The traditional image of a function is a machine, with a slot on one side where numbers go in and a slot on the other side where numbers come out.

$5 \rightarrow$		$\rightarrow 16$

A number goes in. A number comes out. The function is the machine, the process that turns 4 into 14 or 5 into 16 or 100 into 206.

The point of this image is that the function is not the numbers, but the machine itself—the **process**, not the results of the process.

The primary purpose of “The Function Game” that you play on Day 1 is to get across this idea of a numerical process. In this game, one student (the “leader”) is placed in the role of a function. “Whenever someone gives you a number, you double that number, add 6, and give back the result.” It should be very clear, as you perform this role, that you are **not** modeling a number, a variable, or even a list of numbers. You are instead modeling a process—or an algorithm, or a recipe—for turning numbers into other numbers. That is what a function is.

The function game also contains some more esoteric functions: “Respond with –3 no matter what number you are given,” or “Give back the lowest prime number that is greater than or equal to the number you were given.” Students playing the function game often ask “Can a function **do** that?” The answer is always yes (with one caveat mentioned below). So another purpose of the function game is to expand your idea of what a function can do. Any process that consistently turns numbers into other numbers, is a function.

By the way—having defined the word “function” I just want to say something about the word “equation.” An “equation” is when you “equate” two things—that is to say, set them equal. So $x^2 - 3$ is a function, but it is not an equation. $x^2 - 3 = 6$ is an equation. An “equation” always has an equal sign in it.

Function Concepts -- The Rule of Consistency

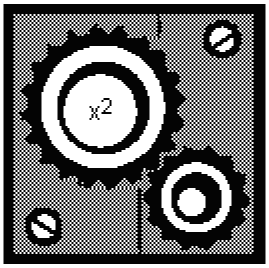
This module discusses how functions must be consistent.

There is only one limitation on what a function can do: **a function must be consistent.**

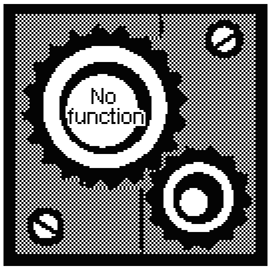
For instance, the function in the above drawing is given a 5, and gives back a 16. That means this particular function turns 5 into 16—always. That particular function can never take in a 5 and give back a 14. This “rule of consistency” is a very important constraint on the nature of functions.

Note:This rule does not treat the inputs and outputs the same!

For instance, consider the function $y = x^2$. This function takes both 3 and -3 and turns them into 9 (two different inputs, same output). That is allowed. However, it is not reversible! If you take a 9 and turn it into both a 3 and a –3 (two different outputs, same input), you are not a function.

$3 \rightarrow$ $-3 \rightarrow$		$\rightarrow 9$

If 3 goes in, 9 comes out. If –3 goes in, 9 also comes out. No problem: x^2 is a function.

$\rightarrow 9$		$3 \rightarrow$ $-3 \rightarrow$

If 9 goes in, both –3 and 3 come out. This **violates the rule of consistency:** no function can do this

This asymmetry has the potential to cause a great deal of confusion, but it is a very important aspect of functions.

Function Concepts -- Four Ways to Represent a Function

This module discusses how it is possible to describe functions in four different ways: graphically, verbally, algebraically, and numerically.

Modern Calculus texts emphasize that a function can be expressed in four different ways.

1. **Verbal** - This is the first way functions are presented in the function game: “Double and add six.”
2. **Algebraic** - This is the most common, most concise, and most powerful representation: $2\mathrm{x}+6$. Note that in an algebraic representation, the input number is represented as a variable (in this case, an x).
3. **Numerical** - This can be done as a list of value pairs, as $(4,14)$ — meaning that if a 4 goes in, a 14 comes out. (You may recognize this as (x,y) points used in graphing.)
4. **Graphical** - This is discussed in detail in the section on graphing.

These are **not** four different types of functions: they are four different views of the same function. One of the most important skills in Algebra is converting a function between these different forms, and this theme will recur in different forms throughout the text.

Function Concepts -- Domain and Range

This module defines the domain and range of a function.

Consider the function $y = \sqrt{x}$. If this function is given a 9 it hands back a 3. If this function is given a 2 it hands back…well, it hands back $\sqrt{2}$, which is approximately 1.4. The answer cannot be specified exactly as a fraction or decimal, but it is a perfectly good answer nonetheless.

On the other hand, what if this function is handed –4? There is no $\sqrt{-4}$, so the function has no number to hand back. If our function is a computer or calculator, it responds with an error message. So we see that this function is able to respond to the numbers 9 and 2, but it is not able to respond in any way to the number -4. Mathematically, we express this by saying that 9 and 2 are in the "domain" of the square root function, and –4 is **not** in the domain of this function.

Domain

> The **domain** of a function is all the numbers that it can successfully act on. Put another way, it is all the numbers that can go into the function.

A square root cannot successfully act on a negative number. We say that "The domain of $\sqrt{x}$ is all numbers
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such that
[missing_resource: graphics2.wmf]

" meaning that if you give this function zero or a positive number, it can act on it; if you give this function a negative number, it cannot.

A subtler example is the function $y = \sqrt{x+7}$. Does this function have the same domain as the previous function? No, it does not. If you hand this function a –4 it successfully hands back $\sqrt{3}$ (about 1.7). –4 is in the domain of this function. On the other hand, if you hand this function a –8 it attempts to take $\sqrt{-1}$ and fails; –8 is **not** in the domain of this function. If you play with a few more numbers, you should be able to convince yourself that the domain of this function is all numbers x such that $x \geq -7$.

You are probably familiar with two mathematical operations that are **not allowed**. The first is, you are not allowed to take the square root of a negative number. As we have seen, this leads to restrictions on the domain of any function that includes square roots.

The second restriction is, you are not allowed to divide by zero. This can also restrict the domain of functions. For instance, the function $y = \frac{1}{x^2-4}$ has as its domain all numbers **except** $x = 2$ and $x = -2$. These two numbers both cause the function to attempt to divide by 0, and hence fail. If you ask a calculator to plug $x = 2$ into this function, you will get an error message.

So: if you are given a function, how can you find its domain? Look for any number that puts a negative number under the square root; these numbers are **not** in the domain. Look for any number that causes the function to divide by zero; these numbers are not in the domain. All other numbers **are** in the domain.

Function	**Domain**	**Comments**
$\sqrt{x}$	$x \geq 0$	You can take the square root of 0, or of any positive number, but you cannot take the square root of a negative number.

Function	Domain	Comments
$\sqrt{x+7}$	$x \geq -7$	If you plug in any number **greater than or equal to –7,** you will be taking a legal square root. If you plug in a number **less than –7**, you will be taking the square root of a negative number.This domain can also be understood graphically: the graph $y=\sqrt{x}$ has been moved 7 units to the left. See “horizontal permutations” below.
$\frac{1}{x}$	$x \neq 0$	In other words, the domain is “all numbers except 0.” You are not allowed to divide by 0. You are allowed to divide by anything else.
$\frac{1}{x-3}$	$x \neq 3$	If $x=3$ then you are dividing by 0, which is **not** allowed. If $x=0$you are dividing by –3, which is allowed. So be careful! The rule is **not** “when you are dividing, x cannot be 0.” The rule is “ x can never be any value that would put a 0 in the denominator.”
$\frac{1}{x^2-4}$	$x \neq \pm 2$	Or, “ $x$can be any number except 2 or –2.” Either of these x values will put a 0 in the denominator, so neither one is allowed.

Function	**Domain**	**Comments**
$2^x + x^2 - 3\text{x} + 4$	All numbers	You can plug any x value into this function and it will come back with a number.
$\frac{\sqrt{x-3}}{x-5}$	$x \geq 3$ $x \neq 5$	In words, the domain is **all numbers greater than or equal to 3, except the number 5**. Numbers less than 3 put negative numbers under the square root; 5 causes a division by 0.

You can confirm all these results with your calculator; try plugging numbers into these functions, and see when you get errors!

A related concept is **range**.

Range

> The **range** of a function is all the numbers that it may possibly produce. Put another way, it is all the numbers that can come **out** of the function.

To illustrate this example, let us return to the function $y = \sqrt{x+7}$. Recall that we said the domain of this function was all numbers x such that $x \geq -7$; in other words, you are allowed to put any number greater than or equal to –7 into this function.

What numbers might come out of this function? If you put in a –7 you get out a 0. ($\sqrt{0} = 0$) If you put in a –6 you get out $\sqrt{1} = 1$. As you increase the x value, the y values also increase. However, if you put in $x = -8$ nothing comes out at all. Hence, the range of this function is all numbers y such that $y \geq 0$. That is, this function is capable of handing back 0 or any positive number, but it will never hand back a negative number.

It’s easy to get the words domain and range confused—and it’s important to keep them distinct, because although they are **related** concepts, they are **different** from each other. One trick that sometimes helps is to remember that, in everyday useage, “your domain” is your home, your land—it is where you begin. A function begins in its own domain. It ends up somewhere out on the range.

A different notation for domain and range

Domains and ranges above are sometimes expressed as intervals, using the following rules:

- Parentheses $()$ mean “an interval starting or ending here, but not including this number”
- Square brackets $[]$ mean “an interval starting or ending here, including this number”

This is easiest to explain with examples.

This notation...	**...means this...**	**...or in other words**
$(-3,5)$	All numbers between –3 and 5, **not including** –3 and 5.	$-3 < x < 5$
$[-3,5]$	All numbers between –3 and 5, **including** –3 and 5.	$-3 \leq x \leq 5$
$[-3,5)$	All numbers between –3 and 5, **including** –3 **but not** 5.	$-3 \leq x < 5$

This notation...	**...means this...**	**...or in other words**
$(-\infty\ 10]$	All numbers less than or equal to 10.	$x \leq 10$
$(23\ \infty)$	All numbers greater than 23.	$x > 23$
$(-\infty\ 4)$ $(4,\infty)$	All numbers less than 4, and all numbers greater than 4. In other words, **all numbers except** 4.	$x \neq 4$

Why are functions so important that they form the heart of math from Algebra II onward?

Functions are used whenever **one variable depends on another variable.** This relationship between two variables is the most important in mathematics. It is a way of saying “If you tell me what x is, I can tell you what y is.” We say that y “depends on” x, or y “is a function of” x.

A few examples:

Example:
Function Concepts -- Functions in the Real World

- "The area of a circle depends on its radius."
- "The amount of money Alice makes depends on the number of hours she works."
- “Max threw a ball. The height of the ball depends on how many seconds it has been in the air.”

In each case, there are two variables. Given enough information about the scenario, you could assert that **if you tell me this variable, I will tell you that one**. For instance, suppose you know that Alice makes $100 per day. Then we could make a chart like this.

If Alice works this many days...	**...she makes this many dollars**

If Alice works this many days...	...she makes this many dollars
0	0
1	100
1½	150
8	800

If you tell me how long she has worked, I will tell you how much money she has made. Her earnings "depend on" how long she works.

The two variables are referred to as the **dependent variable** and the **independent variable**. The dependent variable is said to "depend on" or "be a function of" the independent variable. "The height of the ball is a function of the time."

Example:
Bad Examples of Functional Relationships

- "The number of Trojan soldiers depends on the number of Greek soldiers."
- "The time depends on the height of the ball."

The first of these two examples is by far the most common. It is simply not true. There may be a relationship between these two quantities—for instance, the **sum** of these two variables might be the total number of soldiers, and the **difference** between these two quantities might suggest whether the battle will be a fair one. But there is no **dependency** relationship—that is, no way to say "If you tell me the number of Greek soldiers, I will tell you the number of Trojan soldiers”—so this is not a function.

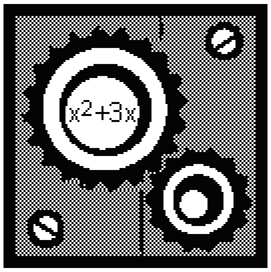
The second example is subtler: it confuses the **dependent** and the **independent** variables. The height depends on the time, not the other way around. More on this in the discussion of “Inverse Functions".

Function Concepts -- Function Notation
This module describes notation for functions.

Function Notation

Functions are represented in math by parentheses. When you write $f(x)$ you indicate that the variable f is a function of—or depends on—the variable x.

For instance, suppose $f(x) = x^2 + 3x$. This means that f is a function that takes whatever you give it, and squares it, and multiplies it by 3, and adds those two quantities.

$7 \rightarrow$ $10 \rightarrow$ $x \rightarrow$ $y \rightarrow$ a dog $\rightarrow$		$\rightarrow f(7) = 7^2 + 3(7) = 70$ $\rightarrow f(10) = 10^2 + 3(10) = 130$ $\rightarrow f(x) = x^2 + 3x$ $\rightarrow f(y) = y^2 + 3y$ $\rightarrow f(\text{dog}) = (\text{dog})^2 + 3(\text{dog})$ (*not in the domain)

The notation $f(7)$ means “plug the number 7 into the function f.” It does **not** indicate that you are multiplying f times 7. To evaluate $f(7)$ you take the function $f(x)$ and replace all occurrences of the variable x with the number 7. If this function is given a 7 it will come out with a 70.

If we write $f(y) = y^2 + 3y$ we have **not specified a different function**. Remember, the function is not the variables or the numbers, it is the process. $f(y) = y^2 + 3y$ also means “whatever number comes in, square it, multiply it by 3, and add those two quantities.” So it is a different way of writing the same function.

Just as many students expect all variables to be named x, many students—and an unfortunate number of parents—expect all functions to be named f. The correct rule is that—whenever possible—**functions, like variables, should be named descriptively**. For instance, if Alice makes $100/day, we might write:

- Let m equal the amount of money Alice has made (measured in dollars)
- Let t equal the amount of time Alice has worked (measured in days)
- Then, $m(t) = 100t$

This last equation should be read " m is a function of t (or m depends on t). Given any value of the variable t, you can multiply it by 100 to find the corresponding value of the variable m."

Of course, this is a very simple function! While simple examples are helpful to illustrate the concept, it is important to realize that very complicated functions are also used to model real world relationships. For instance, in Einstein's Special Theory of Relativity, if an object is going very fast, its mass is multiplied by $\frac{1}{\sqrt{1-\frac{v^2}{9\cdot10^{16}}}}$. While this can look extremely intimidating, it is just another function. The speed v is the independent variable, and the mass m is dependent. Given any speed v you can determine how much the mass m is multiplied by.

Function Concepts -- Algebraic Generalizations

This module discusses the generalizations of algebraic functions and their implications.

When you have a "generalization," you have one **broad fact** that allows you to assume many **specific** facts as examples.

Example:
Generalization: "Things fall down when you drop them."
Specific facts, or examples:

- Leaves fall down when you drop them
- Bricks fall down when you drop them
- Tennis balls fall down when you drop them

If **any one** of the individual statements does not work, the generalization is invalid. (This generalization became problematic with the invention of the helium balloon.)

Scientists tend to work **empirically,** meaning they start with the specific facts and work their way back to the generalization. Generalizations are valued in science because they bring order to apparently disconnected facts, and that order in turn suggests underlying theories.

Mathematicians also spend a great deal of time looking for generalizations. When you have an "algebraic generalization" you have one **algebraic** fact that allows you to assume many **numerical** facts as examples.

Consider, for instance, the first two functions in the function game.

1. Double the number, then add six.
2. Add three to the number, then double.

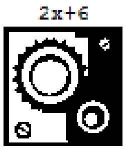
These are very different "recipes." However, their inclusion in the function game is a bit unfair, because—here comes the generalization—**these two functions will always give the same answer.** Whether the input is positive or negative, integer or fraction, small or large, these two functions will mimic each other perfectly. We can express this generalization in words.

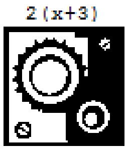
Example:
Generalization: If you plug a number into the function **double and add six**, and plug the same number into the function **add three and double**, the two operations will give the same answer.
Specific facts, or examples:

- If you double –5 and add six; or, if you add –5 to 3 and then double; you end up with the same answer.
- If you double 13 and add six; or, if you add 13 to 3 and then double; you end up with the same answer.

There is literally an infinite number of specific claims that fit this pattern. We don’t need to prove or test each of these claims individually: once we have proven the generalization, we know that **all** these facts must be true.

We can express this same generalization pictorially by showing two “function machines” that always do the same thing.

$-5 \rightarrow$ $0 \rightarrow$ $13 \rightarrow$		$\rightarrow 2(-5)+6=-4$ $\rightarrow 2(0)+6=6$ $\rightarrow 2(13)+6=32$

$-5 \rightarrow$ $0 \rightarrow$ $13 \rightarrow$		$\rightarrow 2(-5)+6=-4$ $\rightarrow 2(0)+6=6$ $\rightarrow 2(13)+6=32$

But the most common way to express this generalization is algebraically, by asserting that these two functions **equal** each other.

Equation:

$$2x + 6 = 2(x + 3)$$

Many beginning Algebra II students will recognize this as the distributive property. Given $2(x + 3)$ they can correctly turn it into $2x + 6$. **But they often fail to realize what this equality means—that given the same input, the two functions will always yield the same output.**

Example:
Generalization: $2x + 6 = 2(x + 3)$
Specific facts, or examples:

- $(2 \times -5) + 6 = 2 \times (-5 + 3)$
- $(2 \times 0) + 6 = 2 \times (0 + 3)$
- $(2 \times 13) + 6 = 2 \times (13 + 3)$

It's worth stopping for a moment here to think about the $=$ symbol. Whenever it is used, $=$ indicates that two things are the same. However, the following two equations use the $=$ in very different ways.

Equation:

$$2x^2 + 5x = 3$$

Equation:

$$\frac{2x^2 - 18}{x + 3} = 2x - 6$$

In the first equation, the $=$ challenges you to **solve for** x. "Find all the x values that make this equation true." The answers in this case are $x = \frac{1}{2}$ and $x = -3$. If you plug in either of these two x-values, you get a true equation; for any other x-value, you get a false equation.

The second equation cannot be solved for x; the $=$ sign in this case is asserting an equality that is true for **any** x-value. Let’s try a few.

Example:
Generalization: $\frac{2\mathrm{x}^2-18}{x+3} = 2\mathrm{x} - 6$
Specific facts, or examples:

$x = 3$	$\frac{2(3)^2-18}{(3)+3} = \frac{18-18}{6} = 0$	$2(3) - 6 = 0$ ✓
$x = -2$	$\frac{2(-2)^2-18}{(-2)+3} = \frac{8-18}{1} = -10$	$2(-2) - 6 = -10$ ✓
$x = 0$	$\frac{2(0)^2-18}{(0)+3} = \frac{0-18}{3} = -6$	$2(0) - 6 = -6$ ✓
$x = \frac{1}{2}$	$\frac{2\left(\frac{1}{2}\right)^2-18}{\left(\frac{1}{2}\right)+3} = \frac{\frac{1}{2}-18}{\frac{7}{2}} = \left(\frac{-35}{2}\right)\left(\frac{2}{7}\right) = -5$	$2\left(\frac{1}{2}\right) - 6 = -5$ ✓

With a calculator, you can attempt more difficult values such as $x = -26$ or $x = \pi$; in every case, the two formulas will give the same answer. When we assert that two very different functions will always produce the same answers, we are making a very powerful generalization.

Exception: $x = -3$ is outside the domain of one of these two functions. In this important sense, the two functions are not in fact equal. Take a moment to make sure you understand why this is true!

Such generalizations are very important because they allow us to **simplify**.

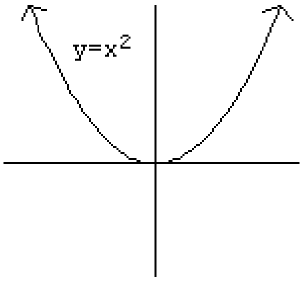
Suppose that you were told “I am going to give you a hundred numbers. For each number I give you, square it, then double the answer, then subtract eighteen, then divide by the original number plus three.” This kind of operation comes up all the time. But you would be quite relieved to discover that you can accomplish the same task by simply doubling each number and subtracting 6! The generalization in this case is $\frac{2\mathrm{x}^2-18}{x+3}=2\mathrm{x}-6$; you will be creating exactly this sort of generalization in the chapter on Rational Expressions.

Function Concepts -- Graphing

This module describes how to graph basic functions.

Graphing, like algebraic generalizations, is a difficult topic because many students know **how to do it** but are not sure **what it means**.

For instance, consider the following graph:



If I asked you “Draw the graph of $y=x^2$” you would probably remember how to plot points and draw the shape.

But suppose I asked you this instead: “Here’s a function, $y=x^2$. And here’s a shape, that sort of looks like a U. What do they actually have to do with each other?” This is a harder question! What does it **mean** to graph a function?

The answer is simple, but it has important implications for a proper understanding of functions. Recall that every point on the plane is designated by a unique (x,y) pair of coordinates: for instance, one point is $(5,3)$. We say that its x -value is 5 and its y -value is 3.

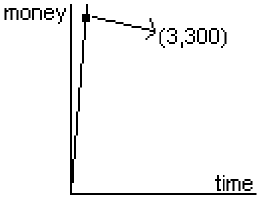
A few of these points have the particular property that their y -values are the square of their x -values. For instance, the points $(0,0)$, $(3,9)$, and $(-5,25)$ all have that property. $(5,3)$ and $(-2,-4)$ do not.

The graph shown—the pseudo-U shape—is **all the points in the plane that have this property**. Any point whose y-value is the square of its x-value is on this shape; any point whose y-value is not the square of its x-value is not on this shape. Hence, glancing at this shape gives us a complete visual picture of the function $y=x^2$ **if we know how to interpret it correctly**.

Graphing Functions

Remember that every function specifies a relationship between two variables. When we graph a function, we put the **independent variable** on the x-axis, and the **dependent variable** on the y-axis.

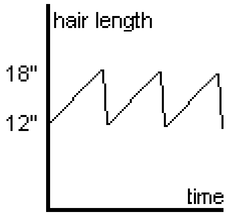
For instance, recall the function that describes Alice's money as a function of her hours worked. Since Alice makes $12/hour, her financial function is $m(t) = 12t$. We can graph it like this.



This simple graph has a great deal to tell us about Alice's job, if we read it correctly.

- The graph contains the point $(3{,}300)$.What does that tell us? That after Alice has worked for three hours, she has made $300.
- The graph goes through the origin (the point $(0{,}0)$). What does that tell us? That when she works 0 hours, Alice makes no money.
- The graph exists only in the first quadrant. What does that tell us? On the mathematical level, it indicates the **domain** of the function ($t \geq 0$) and the **range** of the function ($m \geq 0$). In terms of the situation, it tells us that Alice cannot work negative hours or make negative money.
- The graph is a straight line. What does that tell us? That Alice makes the same amount of money every day: every day, her money goes up by $100. ($100/day is the **slope** of the line—more on this in the section on linear functions.)

Consider now the following, more complicated graph, which represents **Alice's hair length as a function of time** (where time is now measured in weeks instead of hours).



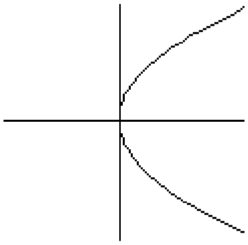
What does this graph $h(t)$ tell us? We can start with the same sort of simple analysis.

- The graph goes through the point $(0{,}12)$.This tells us that at time $(t = 0)$, Alice’s hair is 12" long.
- The range of this graph appears to be $12 \leq h \leq 18$. Alice never allows her hair to be shorter than 12" or longer than 18".

But what about the shape of the graph? The graph shows a gradual incline up to 18", and then a precipitous drop back down to 12"; and this pattern repeats throughout the shown time. The most likely explanation is that Alice’s hair grows slowly until it reaches 18", at which point she goes to the hair stylist and has it cut down, within a very short time (an hour or so), to 12". Then the gradual growth begins again.

The rule of consistency, graphically

Consider the following graph.



This is our earlier “U” shaped graph ($y = x^2$) turned on its side. This might seem like a small change. But ask this question: what is y when $x = 3$? This question has two answers. This graph contains the points $(3, -9)$ and $(3{,}9)$. So when $x = 3$, y is both 9 and –9 on this graph.

This violates the only restriction on functions—the rule of consistency. Remember that the x-axis is the independent variable, the y-axis the dependent. In this case, one “input” value (3) is leading to two different “output” values $(-9,9)$ We can therefore conclude that this graph **does not represent a function at all.** No function, no matter how simple or complicated, could produce this graph.

This idea leads us to the “vertical line test,” the graphical analog of the rule of consistency.

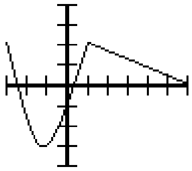
The Vertical Line Test

> If you can draw **any** vertical line that touches a graph in two places, then that graph violates the rule of consistency and therefore does not represent any function.

It is important to understand that the vertical line test is not a new rule! It is the graphical version of the rule of consistency. If any vertical line touches a graph in two places, then the graph has two different y-values for the same x-value, and this is the only thing that functions are not allowed to do.

What happens to the graph, when you add 2 to a function?

Suppose the following is the graph of the function $y=f(x)$.



$y=f(x)$; Contains the following points (among others):

$(-3,2)$,
$(-1,-3)$
, $(1,2)$,
$(6,0)$

We can see from the graph that the domain of the graph is $-3 \leq x \leq 6$ and the range is $-3 \leq y \leq 2$.

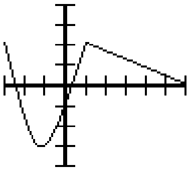
Question: **What does the graph of** $y = f(x) + 2$ **look like**?

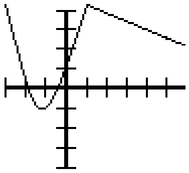
This might seem an impossible question, since we do not even know what the function $f(x)$ is. But we don’t need to know that in order to plot a few points.

x	$f(x)$	$f(x+2)$	**so** $y = f(x)$ **contains this point**	**and** $y = f(x) + 2$ **contains this point**
–3	2	4	$(-3,2)$	$(-3,4)$
–1	–3	–1	$(-1,-3)$	$(-1,-1)$
1	2	4	$(1,2)$	$(1,4)$
6	0	2	$(6,0)$	$(6,2)$

If you plot these points on a graph, the pattern should become clear. Each point on the graph is **moving up by two**. This comes as no surprise: since you added 2 to each y-value, and adding 2 to a y-value moves any point up by 2. So the new graph will look identical to the old, only moved up by 2.

$y = f(x)$ $y = f(x) + 2$; All y-values are 2 higher

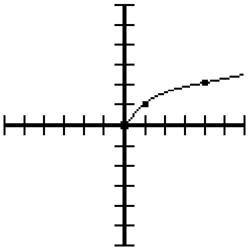


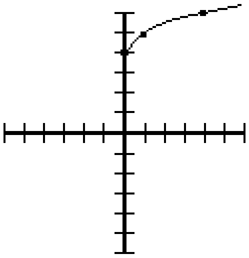


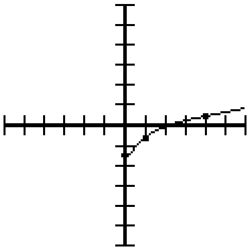
In a similar way, it should be obvious that if you **subtract** 10 from a function, the graph moves **down** by 10. Note that, in either case, the **domain** of the function is the same, but the **range** has changed.

These permutations work for **any function**. Hence, given the graph of the function $y = \sqrt{x}$ below (which you could generate by plotting points), you can produce the other two graphs **without** plotting points, simply by moving the first graph up and down.

$y = \sqrt{x}$ $y = \sqrt{x} + 4$ $y = \sqrt{x} - 1\frac{1}{2}$



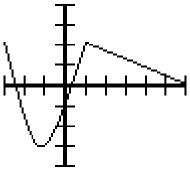




Other vertical permutations

Adding or subtracting a constant from $f(x)$, as described above, is one example of a **vertical permutation:** it moves the graph up and down. There are other examples of vertical permutations.

For instance, what does **doubling** a function do to a graph? Let’s return to our original function:



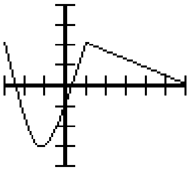
$y = f(x)$

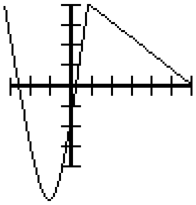
What does the graph $y = 2\text{f}(x)$ look like? We can make a table similar to the one we made before.

x	$f(x)$	$2\text{f}(x)$	**so $y = 2\text{f}(x)$ contains this point**
–3	2	4	$(-3,4)$
–1	–3	–6	$(-1,-6)$
1	2	4	$(1,4)$
6	0	0	$(6,0)$

In general, the high points move higher; the low points move lower. The entire graph is **vertically stretched**, with each point moving farther away from the x-axis.

$y = f(x)$ $y = 2\mathrm{f}(x)$; All y-values are doubled

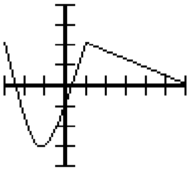


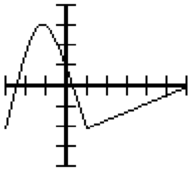


Similarly, $y = \frac{1}{2}f(x)$ yields a graph that is vertically compressed, with each point moving toward the x-axis.

Finally, what does $y = -f(x)$ look like? All the positive values become negative, and the negative values become positive. So, point by point, the entire graph flips over the x-axis.

$y = f(x)$ $y = -\mathrm{f}(x)$; All y-values change sign

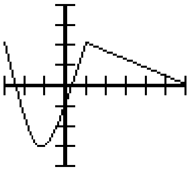




What happens to the graph, when you add 2 to the x value?

Vertical permutations affect the y-value; that is, the output, or the function itself. Horizontal permutations affect the x-value; that is, the numbers that come in. They often do the opposite of what it naturally seems they should.

Let’s return to our original function $y = f(x)$.



$y = f(x)$; Contains the following points (among others): $(-3,2)$, $(-1,-3)$, $(1,2)$, $(6,0)$

Suppose you were asked to graph $y = f(x+2)$. Note that this is **not the same** as $f(x) + 2$! The latter is an instruction to run the function, and then add 2 to all results. But $y = f(x+2)$ is an instruction to add 2 to every x-value **before plugging it into the function.**

- $f(x) + 2$ changes y, and therefore shifts the graph vertically
- $f(x+2)$ changes x, and therefore shifts the graph horizontally.

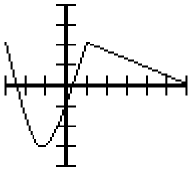
But which way? In analogy to the vertical permutations, you might expect that adding two would shift the graph to the right. But let’s make a table of values again.

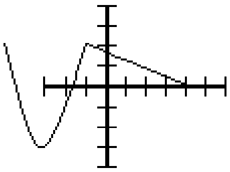
x	$x+2$	$f(x+2)$	**so** $y=f(x+2)$ **contains this point**
–5	–3	f(–3)=2	$(-5,2)$
–3	–1	f(–1)=–3	$(-3,-3)$
–1	1	f(1)=2	$(-1,2)$
4	6	f(6)=0	$(4,0)$

This is a very subtle, very important point—please follow it closely and carefully! First of all, make sure you understand where all the numbers in that table came from. Then look what happened to the original graph.

Note:The original graph $f(x)$ **contains the point** $(6,0)$; **therefore,** $f(x+2)$ **contains the point** $(4,0)$. The point has moved two spaces to the left.

$y=f(x)$ $y=\mathrm{f}(\mathrm{x}+2)$; Each point is shifted to the left





You see what I mean when I say horizontal permutations “often do the opposite of what it naturally seems they should”? **Adding** two moves the graph to the **left**.

Why does it work that way? Here is my favorite way of thinking about it. $f(x-2)$ is an instruction that says to each point, “look two spaces to your left, and copy what the original function is doing **there**.” At $x=5$ it does what $f(x)$ does at $x=3$. At $x=10$, it copies $f(8)$. And so on. Because it is always copying $f(x)$ to its **left**, this graph ends up being a copy of $f(x)$ moved to the **right**. If you understand this way of looking at it, all the rest of the horizontal permutations will make sense.

Of course, as you might expect, subtraction has the opposite effect: $f(x-6)$ takes the original graph and moves it 6 units to the **right**. In either case, these horizontal permutations affect the **domain** of the original function, but not its **range**.

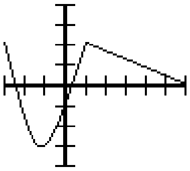
Other horizontal permutations

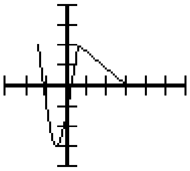
Recall that $y=2\text{f}(x)$ **vertically stretches** a graph; $y=\frac{1}{2}f(x)$ **vertically compresses**. Just as with addition and subtraction, we will find that the horizontal equivalents work backward.

x	2x	$f(2\text{x})$	**so** $y=2\text{f}(x)$ **contains this point**
–1½	–3	2	$(-1\frac{1}{2},2)$
–½	–1	–3	$(-\frac{1}{2};-3)$
½	1	2	$(\frac{1}{2};2)$
3	6	0	$(3{,}0)$

The original graph $f(x)$ contains the point $(6{,}0)$; therefore, $f(2\text{x})$ contains the point $(3{,}0)$. Similarly, $(-1;-3)$ becomes $(-\frac{1}{2};-3)$. Each point is closer to the y-axis; the graph has horizontally **compressed**.

$y = f(x)$ $y = \mathrm{f}(2\mathrm{x})$; Each point is twice as close to the y-axis

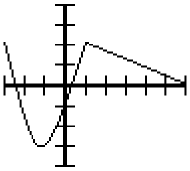


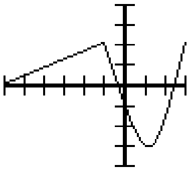


We can explain this the same way we explained $f(x-2)$. In this case, $f(2\mathrm{x})$ is an instruction that says to each point, “Look outward, at the x-value that is double yours, and copy what the original function is doing **there**.” At $x = 5$ it does what $f(x)$ does at $x = 10$. At $x = -3$, it copies $f(-6)$. And so on. Because it is always copying f(x) **outside** itself, this graph ends up being a copy of $f(x)$ moved **inward**; ie a compression. Similarly, $f(\frac{1}{2}x)$ causes each point to look inward toward the y-axis, so it winds up being a **horizontally stretched** version of the original.

Finally, $y = f(-x)$ does precisely what you would expect: it flips the graph around the y-axis. $f(-2)$ is the old $f(2)$ and vice-versa.

$y = f(x)$ $y = \mathrm{f}(\text{-}\mathrm{x})$; Each point flips around the y-axis

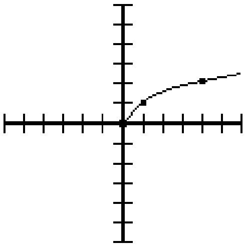




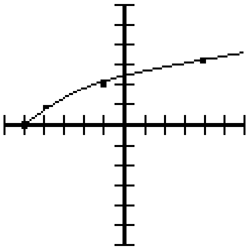
All of these permutations do not need to be memorized: only the general principles need to be understood. But once they are properly understood, even a complex graph such as $y = -2(x+3)^2 + 5$ can be easily graphed. You take the (known) graph of $y = x^2$, flip it over the x-axis (because of the negative sign), stretch it vertically (the 2), move it to the left by 3, and move it up 5.

With a good understanding of permutations, and a very simple list of known graphs, it becomes possible to graph a wide variety of important functions. To complete our look at permutations, let's return to the graph of $y = \sqrt{x}$ in a variety of flavors.

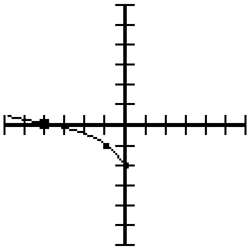
$y = \sqrt{x}$; Generated by plotting points; Contains $(0{,}0)$, $(1{,}1)$, $(4{,}2)$; Domain: $x \geq 0$; Range: $y \geq 0$; Range: $y \geq 0$



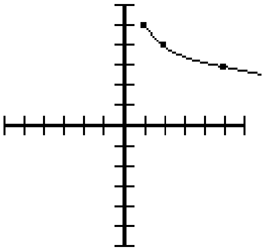
$y = \sqrt{x+5}$; Shifted 5 units to the left; Contains $(-5{,}0)$, $(-4{,}1)$, $(-1{,}2)$; Domain: $x \geq -5$; Range: $y \geq 0$



$y = \sqrt{-x} - 2$; Flipped horizontally, shifted down 2; Contains $(0,-2)$, $(-1,-1)$, $(-4{,}0)$; Domain: $x \leq 0$; Range: $y \geq -2$



$y = -\sqrt{x-1} + 5$; Flipped vertically, shifted 1 to the right and 5 up; Contains $(1{,}5)$, $(2{,}4)$, $(5{,}3)$; Domain: $x \geq 1$; Range: $y \leq 5$



Function Concepts -- Lines

This module discusses lines and their uses, and slope.

Most students entering Algebra II are already familiar with the basic mechanics of graphing lines. Recapping very briefly: the equation for a line is $y = \mathrm{mx} + b$ where b is the y-intercept (the place where the line crosses the y-axis) and m is the slope. If a linear equation is given in another form (for instance, $4\mathrm{x} + 2\mathrm{y} = 5$), the easiest way to graph it is to rewrite it in $y = \mathrm{mx} + b$ form (in this case, $y = -2\mathrm{x} + 2\frac{1}{2}$).

There are two purposes of reintroducing this material in Algebra II. The first is to frame the discussion as **linear functions modeling behavior**. The second is to deepen your understanding of the important concept of slope.

Consider the following examples. Sam is a salesman—he earns a commission for each sale. Alice is a technical support representative—she earns \$100 each day. The chart below shows their bank accounts over the week.

After this many days (t)	**Sam's bank account (S)**	**Alice's bank account (A)**
0 (*what they started with)	\$75	\$750
1	\$275	\$850
2	\$375	\$950
3	\$450	\$1,050
4	\$480	\$1,150

After this many days (t)	**Sam’s bank account (S)**	**Alice’s bank account (A)**
5	$530	$1,250

Sam has some extremely good days (such as the first day, when he made $200) and some extremely bad days (such as the second day, when he made nothing). Alice makes exactly $100 every day.

Let d be the number of days, S be the number of dollars Sam has made, and A be the number of dollars Alice has made. Both S and A are functions of time. But $s(t)$ is **not a linear function**, and $A(t)$**is a linear function**.

Linear Function

> A function is said to be “linear” if **every time the independent variable increases by 1, the dependent variable increases or decreases by the same amount**.

Once you know that Alice’s bank account function is linear, there are only two things you need to know before you can predict her bank account on any given day.

- How much money she started with ($750 in this example). This is called the y-**intercept**.
- How much she makes each day ($100 in this example). This is called the **slope**.

y-intercept is relatively easy to understand. Verbally, it is where the function starts; graphically, it is where the line crosses the y-axis.

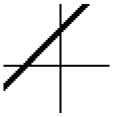
But what about slope? One of the best ways to understand the idea of slope is to convince yourself that all of the following definitions of slope are actually the same.

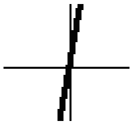
Definitions of Slope		
In our example	**In general**	**On a graph**
Each day, Alice’s bank account increases by 100. So the slope is 100.	Each time the independent variable increases by 1, the dependent variable increases by the slope.	Each time you move to the **right** by 1, the graph goes **up** by the slope.
Between days 2 and 5, Alice earns $300 in 3 days. 300/3=100.Between days 1 and 3, she earns $200 in 2 days. 200/2=100.	Take any two points. The change in the dependent variable, **divided** by the change in the independent variable, is the slope.	Take any two points. The change in y divided by the change in x is the slope. This is often written as $\frac{\Delta y}{\Delta x}$, or as $\frac{\text{rise}}{\text{run}}$
The higher the slope, the faster Alice is making moey.	The higher the slope, the faster the dependent variable increases.	The higher the slope, the faster the graph rises as you move to the right.

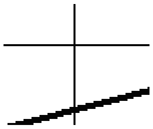
So slope does not tell you where a graph is, but how quickly it is rising. Looking at a graph, you can get an approximate feeling for its slope without any numbers. Examples are given below.

A slope of 1: each time you go **over** 1,	A steep slope of perhaps 3 or 4	A gentle slope of perhaps $\frac{1}{2}$.	A horizontal line has a slope of 0: each time you go	This goes **down** as you move left to right. So the slope is negative. It is

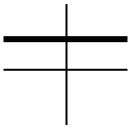
you also
go **up** 1



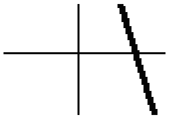




over 1, you
don’t go **up**
at all!



steep: maybe
a –2.



Function Concepts -- Composite Functions

You are working in the school cafeteria, making peanut butter sandwiches for today’s lunch.

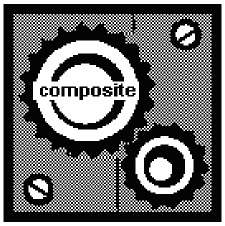
- The more classes the school has, the more children there are.
- The more children there are, the more sandwiches you have to make.
- The more sandwiches you have to make, the more pounds (lbs) of peanut butter you will use.
- The more peanut butter you use, the more money you need to budget for peanut butter.

...and so on. Each sentence in this little story is a function. Mathematically, if c is the number of classes and h is the number of children, then the first sentence asserts the existence of a function $h(c)$.

The principal walks up to you at the beginning of the year and says “We’re considering expanding the school. If we expand to 70 classes, how much money do we need to budget? What if we expand to 75? How about 80?” For each of these numbers, you have to calculate each number from the previous one, until you find the final budget number.

# Classes		# Children		# Sandwiches		lb.		$$

But going through this process each time is tedious. What you want is one function that puts the entire chain together: “You tell me the number of classes, and I will tell you the budget.”

# Classes		$$

This is a **composite function**—a function that represents in one function, the results of an entire **chain of dependent functions**. Since such chains are very common in real life, finding composite functions is a very important skill.

How do you make a composite Function?

We can consider how to build composite functions into the function game that we played on the first day. Suppose Susan takes any number you give her, quadruples it, and adds 6. Al takes any number you give him and divides it by 2. Mathematically, we can represent the two functions like this:

Equation:

$$S(x)=4\text{x}+6$$

Equation:

$$A(x)=\frac{x}{2}$$

To create a chain like the one above, we give a number to Susan; she acts on it, and gives the resulting number to Al; and he then acts on it and hands back a third number.

$3\rightarrow\text{Susan}\rightarrow S(3)=18\rightarrow\text{Al}\rightarrow A(18)=9$

In this example, we are plugging $S(3)$—in other words, 18— into Al's function. In general, for any x that comes in, we are plugging $S(x)$ **into** $A(x)$. So we could represent the entire process as $A(S(x))$. This notation for composite functions is really nothing new: it means that you are plugging $S(x)$ into the A function.

But in this case, recall that $S(x)=4\text{x}+6$. So we can write:

Equation:

$$A(S(x))=\frac{S(x)}{2}=\frac{4\text{x}+6}{2}=2\text{x}+3$$

What happened? We've just discovered a shortcut for the entire process. When you perform the operation $A(S(x))$—that is, when you perform the Al function on the result of the Susan function—you are, in effect, doubling and adding 3. For instance, we saw earlier that when we started with a 3, we ended with a 9. Our composite function does this in one step:

$3\rightarrow 2\text{x}+3\rightarrow 9$

Understanding the meaning of composite functions requires real thought. It requires understanding the idea that this variable depends on that variable, which in turn depends on the other variable; and how that idea is translated into mathematics. **Finding** composite functions, on the other hand, is a purely mechanical process—it requires practice, but no creativity. Whenever you are asked for $f(g(x))$, just plug the $g(x)$ function into the $f(x)$ function and then simplify.

Example:
Building and Testing a Composite Function
$f(x) = x^2 - 4\text{x}$
$g(x) = x + 2$
What is $f(g(x))$?

- To find the composite, plug $g(x)$ into $f(x)$, just as you would with any number.

$$f(g(x)) = (x + 2)^2 - 4(x + 2)$$

- Then simplify.

$$f(g(x)) = (x^2 + 4\text{x} + 4) - (4\text{x} + 8)$$
$$f(g(x)) = x^2 - 4$$

- Let’s test it. $f(g(x))$ means do g, then f. What happens if we start with $x = 9$?

$$7 \rightarrow g(x) \rightarrow 7 + 2 = 9 \rightarrow f(x) \rightarrow (9)^2 - 4(9) = 45$$

- So, if it worked, our **composite function** should do all of that in one step.

$7 \rightarrow x^2 - 4 = (7)^2 - 4 = 45$ It worked!

There is a different notation that is sometimes used for composite functions. This book will consistently use $f(g(x))$ which very naturally conveys the idea of “plugging $g(x)$ into $f(x)$.” However, you will sometimes see the same thing written as $f\,^{\circ}g(x)$, which more naturally conveys the idea of “doing one function, and then the other, in sequence.” The two notations mean the same thing.

Function Concepts -- Inverse Functions

This module describes what inverse functions are and how they can be used.

Let's go back to Alice, who makes $100/day. We know how to answer questions such as "After 3 days, how much money has she made?" We use the function $m(t) = 100t$.

But suppose I want to ask the reverse question: “If Alice has made $300, how many hours has she worked?” This is the job of an inverse function. It gives the same relationship, but reverses the dependent and independent variables. $t(m) = m/100$. Given any amount of money, divide it by 100 to find how many days she has worked.

- **If a function answers the question:** “Alice worked this long, how much money has she made?” **then its inverse answers the question:** “Alice made this much money, how long did she work?"
- **If a function answers the question:** “I have this many spoons, how much do they weigh?” **then its inverse answers the question:** “My spoons weigh this much, how many do I have?”
- **If a function answers the question:** “How many hours of music fit on 12 CDs?” **then its inverse answers the question:** “How many CDs do you need for 3 hours of music?”

How do you recognize an inverse function?

Let’s look at the two functions above:

Equation:

$$m(t) = 100t$$

Equation:

$$t(m) = m/100$$

Mathematically, you can recognize these as inverse functions because they **reverse the inputs and the outputs**.

$3 \rightarrow m(t) = 100t \rightarrow 300$
$300 \rightarrow t(m) = m/100 \rightarrow 3$
✓ Inverse functions

Of course, this makes logical sense. The first line above says that “If Alice works 3 hours, she makes $300.” The second line says “If Alice made $300, she worked 3 hours.” It’s the same statement, made in two different ways.

But this “reversal” property gives us a way to test any two functions to see if they are inverses. For instance, consider the two functions:

Equation:

$$f(x) = 3\text{x} + 7$$

Equation:

$$g(x) = \frac{1}{3}x - 7$$

They **look** like inverses, don’t they? But let’s test and find out.

$2 \rightarrow 3\text{x} + 7 \rightarrow 13$
$13 \rightarrow \frac{3}{\text{x}} - 7 \rightarrow \frac{13}{3} - 7 \rightarrow -\frac{8}{3}$
✗ Not inverse functions

The first function turns a 2 into a 13. But the second function does **not** turn 13 into 2. So these are not inverses.

On the other hand, consider:
Equation:

$$f(x) = 3\text{x} + 7$$

Equation:

$$g(x) = \frac{1}{3}(x - 7)$$

Let’s run our test of inverses on these two functions.

$2 \rightarrow 3\text{x} + 7 \rightarrow 13$
$13 \rightarrow \frac{1}{3}(x - 7) \rightarrow 2$
✓ Inverse functions

So we can see that these functions do, in fact, reverse each other: they are inverses.

A common example is the Celsius-to-Fahrenheit conversion:
Equation:

$$F(C) = \left(\frac{9}{5}\right)C + 32$$

Equation:

$$C(F) = \left(\frac{5}{9}\right)(F - 32)$$

where C is the Celsius temperature and F the Fahrenheit. If you plug $100^\circ C$ into the first equation, you find that it is $212^\circ F$. If you ask the second equation about $212^\circ F$, it of course converts that back into $100^\circ C$.

The notation and definition of an inverse function

The notation for the inverse function of $f(x)$ is $f^{-1}(x)$. This notation can cause considerable confusion, because it **looks like** an exponent, but it isn't. $f^{-1}(x)$ simply means "the inverse function of $f(x)$." It is defined formally by the fact that if you plug any number x into one function, and then plug the result into the other function, you get back where you started. (Take a moment to convince yourself that this is the same definition I gave above more informally.) We can represent this as a composition function by saying that $f(f^{-1}(x)) = x$.

Inverse Function

> $f^{-1}(x)$ is defined as the **inverse function** of $f(x)$ if it consistently reverses the $f(x)$ process. That is, if $f(x)$ turns a into b, then $f^{-1}(x)$ must turn b into a. More concisely and formally, $f^{-1}(x)$ is the inverse function of $f(x)$ if $f(f^{-1}(x)) = x$.

Finding an inverse function

In examples above, we saw that if $f(x) = 3\text{x} + 7$, then $f^{-1}(x) = \frac{1}{3}(x - 7)$. We also saw that the function $\frac{1}{3}x - 7$, which may have looked just as likely, did **not** work as an inverse function. So in general, given a function, how do you find its inverse function?

Remember that an inverse function reverses the inputs and outputs. When we graph functions, we always represent the incoming number as x and the outgoing number as y. So to find the inverse function, **switch the x and y values, and then solve for** y.

Example:
Building and Testing an Inverse Function

1. Find the inverse function of $f(x) = \frac{2x-3}{5}$
   - **a.**Write the function as $y = \frac{2x-3}{5}$
   - **b.**Switch the x and y variables. $x = \frac{2y-3}{5}$
   - **c.**Solve for y. $5x = 2y - 3$. $5x + 3 = 2y$. $\frac{5x+3}{2} = y$. So $f^{-1}(x) = \frac{5x+3}{2}$.
2. Test to make sure this solution fills the definition of an inverse function.
   - **a.**Pick a number, and plug it into the original function. $9 \rightarrow f(x) \rightarrow 3$.
   - **b.**See if the inverse function reverses this process. $3 \rightarrow f^{-1}(x) \rightarrow 9$. ✓ It worked!

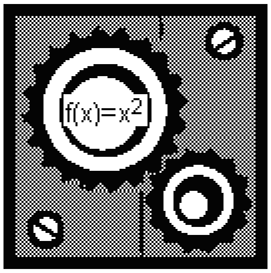
Were you surprised by the answer? At first glance, it seems that the numbers in the original function (the 2, 3, and 5) have been rearranged almost at random.

But with more thought, the solution becomes very intuitive. The original function $f(x)$ described the following process: **double a number, then subtract 3, then divide by 5**. To reverse this process, we need to reverse each step in order: **multiply by 5, then add 3, then divide by 2**. This is just what the inverse function does.

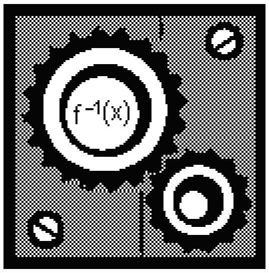
Some functions have no inverse function

Some functions have no inverse function. The reason is the rule of consistency.

For instance, consider the function $y = x^2$. This function takes both 3 and –3 and turns them into 9. No problem: a function is allowed to turn different **inputs** into the same **output**. However, what does that say about the inverse of this particular function? In order to fulfill the requirement of an inverse function, it would have to take 9, and turn it into both 3 and –3—which is the one and only thing that functions are **not** allowed to do. Hence, the inverse of this function would not be a function at all!

$3 \rightarrow$ $-3 \rightarrow$		$9 \rightarrow$ $9 \rightarrow$

If 3 goes in, 9 comes out. If –3 goes in, 9 also comes out. No problem:

$9 \rightarrow$ $9 \rightarrow$		$3 \rightarrow$ $-3 \rightarrow$

But its inverse would have to turn 9 into both 3 and –3. No function can do this, so there is no inverse.

In general, any function that turns multiple inputs into the same output, does not have an inverse function.

What does that mean in the real world? If we can convert Fahrenheit to Celsius, we must be able to convert Celsius to Fahrenheit. If we can ask “How much money did Alice make in 3 days?” we must surely be able to ask “How long did it take Alice to make $500?” When would you have a function that cannot be inverted?

Let’s go back to this example:

Recall the example that was used earlier: “Max threw a ball. The height of the ball depends on how many seconds it has been in the air.” The two variables here are h (the height of the ball) and t (the number of seconds it has been in the air). The function $h(t)$ enables us to answer questions such as “After 3 seconds, where is the ball?”

The inverse question would be “At what time was the ball 10 feet in the air?” The problem with that question is, it may well have **two answers**!

The ball is here...	...after this much time has elapsed
10 ft	2 seconds (*on the way up)
10 ft	5 seconds (*on the way back down)

So what does that mean? Does it mean we can’t ask that question? Of course not. We can ask that question, and we can expect to mathematically find the answer, or answers—and we will do so in the quadratic chapter. However, it does mean that **time is not a function of height** because such a “function” would not be consistent: one question would produce multiple answers.

Quadratic Concepts -- Multiplying Binomials

This module teaches about multiplying binomials. Specifically about common patterns that can be memorized and using the "FOIL" method.

The following three formulae should be memorized.

Equation:

$$(x+a)^2 = x^2 + 2\text{ax} + a^2$$

Equation:

$$(x-a)^2 = x^2 - 2\text{ax} + a^2$$

Equation:

$$(x+a)(x-a) = x^2 - a^2$$

It is important to have these three formulae on the top of your head. It is also nice to be able to show **why** these formulae work, for instance by using FOIL. But the most important thing of all is knowing **what these three formulae mean, and how to use them**.

These three are all “algebraic generalizations,” as discussed in the first unit on functions. That is, they are equations that hold true for **any** values of x and a. It may help if you think of the second equation above as standing for:

$(\text{Anthing} - \text{Anything Else})^2 = \text{Anything}^2 - 2(\text{Anything Else})^2$For instance, suppose the **Anything** (or x) is 5, and the **Anything Else** (or a) is 3.

Example:

$(x-a)^2 = x^2 - 2\text{ax} + a^2$, **when** $x = 5$, $a = 3$.

- $5 - 3^2 \overset{?}{=} 5^2 - 2(3)(5) + 3^2$
- $2^2 \overset{?}{=} 25 - 30 + 9$
- $4 = 4$

It worked! Now, let’s leave the **Anything** as x, but play with different values of a.

Example:

More examples of $(x-a)^2 = x^2 - 2\text{ax} + a^2$

Equation:

$$\begin{array}{rccc}
a = 1: & (x-1)^2 & = & x^2 - 2x + 1 \\
a = 2: & (x-2)^2 & = & x^2 - 4x + 4 \\
a = 3: & (x-3)^2 & = & x^2 - 6x + 9 \\
a = 5: & (x-5)^2 & = & x^2 - 10x + 25 \\
a = 10: & (x-10)^2 & = & x^2 - 20x + 100
\end{array}$$

Once you’ve seen a few of these, the pattern becomes evident: the number **doubles** to create the middle term (the coefficient of x), and **squares** to create the final term (the number).

The hardest thing about this formula is **remembering to use it**. For instance, suppose you are asked to expand:

Equation:

$$(2\mathrm{y} - 6)^2$$

There are three ways you can approach this.

$(2\mathrm{y}-6)^2$, **computed three different ways**		
Square each term	FOIL	Using the formula above
$(2\mathrm{y}-6)^2$ $(2\mathrm{y})^2 - 2(6)(2\mathrm{y}) + 6^2$ $4\mathrm{y}^2 - 24y + 36$	$(2\mathrm{y}-6)(2\mathrm{y}-6)$ $(2\mathrm{y})(2\mathrm{y}) - (2\mathrm{y})6 - (2\mathrm{y})6 + 36$ $4\mathrm{y}^2 - 12y - 12y + 36$ $4\mathrm{y}^2 - 24y + 36$	$(2\mathrm{y}-6)^2$ $(2\mathrm{y})^2 - 2(6)(2\mathrm{y}) + 6^2$ $4\mathrm{y}^2 - 24y + 36$

Did it work? If a formula is true, it should work for any y -value; let’s test each one with $y = 5$. (Note that the second two methods got the same answer, so we only need to test that once.)

$(2\text{y}-6)^2 \stackrel{?}{=} 4\text{y}^2-36$	$(2\text{y}-6)^2 \stackrel{?}{=} 4\text{y}^2-24y+36$
$(2\cdot 5-6)^2 \stackrel{?}{=} 4\text{y}^2-36$	$(2\cdot 5-6)^2 \stackrel{?}{=} 4(5)^2-24\cdot 5+36$
$(10-6)^2 \stackrel{?}{=} 100-36$	$(10-6)^2 \stackrel{?}{=} 100-120+36$
$4^2 \stackrel{?}{=} 64$✗	$4^2 \stackrel{?}{=} 16$✓

We conclude that squaring each term individually does not work. The other two methods both give the same answer, which works.

The first method is the easiest, of course. And it **looks** good. $(2\text{y})^2$ is indeed 4y^2. And 6^2 is indeed 36. But as you can see, it led us to a **false answer**—an algebraic generalization that did not hold up.

I just can't stress this point enough. It sounds like a detail, but it causes errors all through Algebra II and beyond. **When you're adding or subtracting things, and then squaring them, you can't just square them one at a time.** Mathematically, $(x+a)^2 \neq x^2+a^2$. You can confirm this with numbers all day. $(7+3)^2=100$, but $7^2+3^2=58$. They're not the same.

So that leaves the other two methods. FOIL will never lead you astray. But the third approach, the formula, has three distinct advantages.

1. The formula is faster than FOIL.
2. Using these formulae is a specific case of the vital mathematical skill of using **any** formula—learning how to plug numbers and variables into some equation that you've been given, and therefore understanding the abstraction that formulae represent.
3. Before this unit is done, we will be completing the square, which requires running that particular formula **backward**—which you cannot do with FOIL.

Quadratic Concepts -- Factoring
This module discusses how to solve quadratic equations by factoring.

When we multiply, we put things together: when we factor, we pull things apart. Factoring is a critical skill in simplifying functions and solving equations.

There are four basic types of factoring. In each case, I will start by showing a multiplication problem—then I will show how to use factoring to **reverse** the results of that multiplication.

“Pulling Out” Common Factors

This type of factoring is based on the **distributive property**, which (as you know) tells us that:
Equation:

$$2\mathrm{x}\left(4\mathrm{x}^2 - 7\mathrm{x} + 3\right) = 8\mathrm{x}^3 - 14x^2 + 6\mathrm{x}$$

When we factor, we do that in reverse. So we would start with an expression such as $8\mathrm{x}^3 - 14x^2 + 6\mathrm{x}$ and say “Hey, **every one** of those terms is divisible by 2. Also, **every one** of those terms is divisible by x. So we “factor out,” or “pull out,” a 2x.
Equation:

$$8\mathrm{x}^3 - 14x^2 + 6\mathrm{x} = 2\mathrm{x}(\_\_ - \_\_ + \_\_)$$

For each term, we see what happens when we **divide** that term by 2x. For instance, if we divide $8\mathrm{x}^3$ by 2x the answer is $4\mathrm{x}^2$. Doing this process for each term, we end up with:
Equation:

$$8\mathrm{x}^3 - 14x^2 + 6\mathrm{x} = 2\mathrm{x}\left(4\mathrm{x}^2 - 7\mathrm{x} + 3\right)$$

As you can see, this is just what we started with, but in reverse. However, for many types of problems, this factored form is easier to work with.

As another example, consider $6\mathrm{x}+3$. The common factor in this case is 3. When we factor a 3 out of the $6\mathrm{x}$, we are left with $2\mathrm{x}$. When we factor a 3 out of the 3, we are left with...what? Nothing? No, we are left with 1, since we are **dividing** by 3.
Equation:

$$6\mathrm{x}+3=3(2\mathrm{x}+1)$$

There are two key points to take away about this kind of factoring.

1. This is the simplest kind of factoring. Whenever you are trying to factor a complicated expression, **always begin** by looking for **common factors** that you can pull out.
2. A common factor must be common to **all** the terms. For instance, $8\mathrm{x}^3-14x^2+6\mathrm{x}+7$ has no common factor, since the last term is not divisible by either 2 or x.

Factoring Perfect Squares

The second type of factoring is based on the “squaring” formulae that we started with:
Equation:

$$(x+a)^2=x^2+2\mathrm{ax}+a^2$$

Equation:

$$(x-a)^2=x^2-2\mathrm{ax}+a^2$$

For instance, if we see $x^2+6\mathrm{x}+9$, we may recognize the signature of the first formula: the middle term is three **doubled**, and the last term is three

squared. So this is $(x+3)^2$. Once you get used to looking for this pattern, it is easy to spot.
Equation:

$$x^2 + 10x + 25 = (x+5)^2$$

Equation:

$$x^2 + 2\text{x} + 1 = (x+1)^2$$

And so on. If the middle term is **negative**, then we have the second formula:
Equation:

$$x^2 - 8\text{x} + 16 = (x-4)^2$$

Equation:

$$x^2 - 14x + 49 = (x-7)^2$$

This type of factoring only works if you have **exactly this case**: the middle number is something **doubled**, and the last number is that same something **squared**. Furthermore, although the **middle** term can be either positive or negative (as we have seen), the **last** term cannot be negative.

All this may make it seem like such a special case that it is not even worth bothering about. But as you will see with "completing the square" later in this unit, this method is very general, because even if an expression does not look like a perfect square, you can usually **make** it look like one if you want to—and if you know how to spot the pattern.

The Difference Between Two Squares

The third type of factoring is based on the third of our basic formulae:
Equation:

$$(x+a)(x-a)=x^2-a^2$$

You can run this formula in reverse whenever you are **subtracting two perfect squares**. For instance, if we see x^2-25, we recognize that both x^2 and 25 are perfect squares. We can therefore factor it as $(x+5)(x-5)$. Other examples include:

Example:

- $x^2-64=(x+8)(x-8)$
- $16y^2-49=(4\text{y}+7)(4\text{y}-7)$
- $2\text{x}^2-18=2\left(x^2-9\right)=2(x+3)(x-3)$

And so on. Note that, in the last example, we begin by **pulling out** a 2, and we are then left with two perfect squares. This is an example of the rule that you should always **begin by pulling out common factors** before you try anything else!

It is also important to note that you cannot factor the sum of two squares. x^2+4 is a perfectly good function, but it cannot be factored.

Brute Force, Old-Fashioned, Bare-Knuckle, No-Holds-Barred Factoring

In this case, the multiplication that we are reversing is just FOIL. For instance, consider:

Equation:

$$(x+3)(x+7)=x^2+3\text{x}+7\text{x}+21=x^2+10x+21$$

What happened? The 3 and 7 **added** to yield the middle term (10), and **multiplied** to yield the final term (21). We can generalize this as: $(x+a)(x+b)=x^2+(a+b)x+\mathrm{ab}.$

The point is, if you are given a problem such as $x^2+10x+21$ to factor, you look for two numbers that **add** up to 10, and **multiply** to 21. And how do you find them? There are a lot of pairs of numbers that add up to 10, but relatively few that multiply to 21. So you start by looking for factors of 21.

Below is a series of examples. Each example showcases a different aspect of the factoring process, so I would encourage you not to skip over any of them: try each problem yourself, then take a look at what I did.

If you are uncomfortable with factoring, the best practice you can get is to multiply things out. In each case, look at the final answer I arrive at, and multiply it with FOIL. See that you get the problem I started with. Then look back at the steps I took and see how they led me to that answer. The steps will make a lot more sense if you have done the multiplication already.

Exercise:

Problem: Factor $x^2+11x+18$

$(x+\_\_)(x+\_\_)$

Solution:

What **multiplies** to 18? $1\cdot18$, or $2\cdot9$, or $3\cdot6$.

Which of those **adds** to 11? $2+9$.

$(x+2)(x+9)$

Note: Start by listing all factors of the third term. Then see which ones add to give you the middle term you want.

Exercise:

Problem: Factor $x^2 - 13x + 12$

$(x + \_\_)(x + \_\_)$

Solution:

What **multiplies** to 12? $1 \cdot 12$, or $2 \cdot 6$, or $3 \cdot 4$

Which of those **adds** to 13? $1 + 12$

$(x - 1)(x - 12)$

Note: If the middle term is negative, it doesn't change much: it just makes both numbers negative. If this had been $x^2 + 13x + 12$, the process would have been the same, and the answer would have been $(x + 1)(x + 12)$.

Exercise:

Problem: Factor $x^2 + 12x + 24$

$(x + \_\_)(x + \_\_)$

Solution:

What **multiplies** to 24? $1 \cdot 24$, or $2 \cdot 12$, or $3 \cdot 8$, or $4 \cdot 6$

Which of those **adds** to 12? None of them.

It can't be factored. It is "prime."

Note: Some things can’t be factored. Many students spend a long time fighting with such problems, but it really doesn’t have to take long. Try all the possibilities, and if none of them works, it can’t be factored.

Exercise:

Problem: Factor $x^2 + 2\text{x} - 15$

$(x + \_\_)(x + \_\_)$

Solution:

What **multiplies** to 15? $1 \cdot 15$, or $\mathbf{3 \cdot 5}$

Which of those **subtracts** to 2? 5–3

$(x + 5)(x - 3)$

Note: If the **last** term is negative, that changes things! In order to **multiply** to –15, the two numbers will have to have different signs—one negative, one positive—which means they will **subtract** to give the middle term. Note that if the **middle** term were negative, that wouldn’t change the process: the final answer would be reversed, $(x + 5)(x - 3)$. This fits the rule that we saw earlier—changing the sign of the middle term changes the answer a bit, but not the process.

Exercise:

Problem: Factor $2\text{x}^2 + 24x + 72$

Solution:

$2\left(x^2+12x+36\right)$

$2(x+6)^2$

Note: Never forget, **always** start by looking for common factors to pull out. Then look to see if it fits one of our formulae. Only after trying all that do you begin the FOIL approach.

Exercise:

Problem: Factor $3\mathrm{x}^2+14x+16$

$(3\mathrm{x}+\_\_)(x+\_\_)$

Solution:

What **multiplies** to 16? $1\cdot 16$, or $2\cdot 8$, or $4\cdot 4$

Which of those adds to 14 **after tripling one number**? $8+3\cdot 2$

$(3\mathrm{x}+8)(x+2)$

Note: If the x^2 has a coefficient, and if you can’t pull it out, the problem is trickier. In this case, we know that the factored form will look like $(3\mathrm{x}+\_\_)(x+\_\_)$ so we can see that, when we multiply it back, one of those numbers—the one on the right—will be tripled, before they add up to the middle term! So you have to check the number pairs to see if any work that way.

Checking Your Answers

There are two different ways to check your answer after factoring: multiplying back, and trying numbers.

Example:

1. **Problem**: Factor $40x^3 - 250x$
   - $10x(4\text{x} - 25)$ **First, pull out the common factor**
   - $10x(2\text{x} + 5)(2\text{x} - 5)$ **Difference between two squares**
2. **So, does $40x^3 - 250x = 10x(2\text{x} + 5)(2\text{x} - 5)$? First let's check by multiplying back.**
   - $10x(2\text{x} + 5)(2\text{x} - 5)$
   - $= \left(20x^2 + 50x\right)(2\text{x} - 5)$ **Distributive property**
   - $= 40x^3 - 100x^2 + 100x^2 - 250x$ **FOIL**
   - $= 40x^3 - 250x$ ✓
3. **Check by trying a number. This should work for any number. I'll use $x = 7$ and a calculator.**
   - $40x^3 - 250x = 10x(2\text{x} + 5)(2\text{x} - 5)$
   - $40(7)^3 - 250(7) = 10(7)(2 \cdot 7 + 5)(2 \cdot 7 - 5)$
   - $11970 = 11970$ ✓

I stress these methods of checking answers, not just because checking answers is a generally good idea, but because they reinforce key concepts. The first method reinforces the idea that **factoring is multiplication done backward**. The second method reinforces the idea of algebraic generalizations.

Quadratic Concepts -- Solving Quadratic Equations by Factoring

Consider the equation $4x^2 + 14x - 60 = 0$. This is **not** an algebraic generalization, but an equation to be solved for x: that is, it asks the question “What x value, or values, will make this equation true?” We will be solving such equations in three different ways. The fastest and easiest is by factoring.

Using the techniques discussed above, we can rewrite this problem as follows. (Try it for yourself!)

$4x^2 + 14x - 60 = 0$ **Original form**

$2(2x - 5)(x + 6 = 0)$ **Factored form**

The second form may look more complicated than what we started with. But consider what this equation says. There are three numbers: 2, $2x - 5$, and $x + 6$. The equation says that when you multiply these three numbers, you get 0. Ask yourself this crucial question: **How can you multiply numbers and get the answer 0**?

The only way it can happen is if one of the numbers is 0. Take a moment to convince yourself of this: if several numbers **multiply** to give 0, one of those numbers must be 0.

So we have three possibilities.

$2 = 0$	$2x - 5 = 0$	$x + 6 = 0$
(it just isn’t)	$x = 2\frac{1}{2}$	$x = -6$

The moral of the story is: when a quadratic equation is factored, it can be solved easily. In this case, the equation $4\mathrm{x}^2 + 14x - 60 = 0$ has two valid solutions, $x = 2\frac{1}{2}$ and $x = -6$.

Consider this example:
Equation:

$$x^2 - 9\mathrm{x} + 20 = 6$$

A common mistake is to solve it like this.

Example:
$x^2 - 9\mathrm{x} + 20 = 6$, solved **incorrectly**

- $(x-4)(x-5) = 6$
- $(x-4) = 6$
    - $x = 10$ ✗
- $(x-5) = 6$
    - $x = 11$ ✗

All looks good, doesn’t it? The factoring was correct. But if you try $x = 10$ or $x = 11$ in the original equation, you will find that neither one works. What went wrong?

The factoring was correct, but the next step was wrong. Just because $(x-4)(x-5) = 6$ does not mean that either $(x-4)$ or $(x-5)$ has to be 6. There are lots of ways for two numbers to multiply to give 6. This trick only works for 0!

Example:
$x^2 - 9\text{x} + 20 = 6$, solved **correctly**

- $x^2 - 9\text{x} + 14 = 0$
- $(x-7)(x-2) = 0$
- $(x-7) = 0$
    - $x = 7$
- $(x-2) = 0$
    - $x = 2$

You may want to confirm for yourself that these are the correct solutions.

Moral: When solving quadratic equations, always begin by moving **everything to one side of the equation**, leaving only a 0 on the other side. This is true regardless of which of the three methods you use.

Example:
$x^2 + 14x + 49 = 0$

- $(x+7)^2 = 0$
- $x = -7$

Moral: If the left side factors as a perfect square, the quadratic equation has only one solution.

Not all quadratic functions can be factored. This does not mean they have no solutions! If the function cannot be factored, we must use other means to find the solutions.

Quadratic Concepts -- Solving Quadratic Equations by Completing the Square

This module teaches the method of solving quadratic equations by completing the square.

Consider the equation:

Equation:

$$(x+3)^2 = 16$$

We can solve this by analogy to the way that we approached absolute value problems. **something** squared is 16. So what could the **something** be? It could be 4. It could also be -4. So the solution is:

- $x+3=4$
  - $x=1$
- $x+3=-4$
  - $x=-7$

These are the two solutions.

This simple problem leads to a completely general way of solving quadratic equations—because any quadratic equation can be put in a form like the above equation. The key is **completing the square** which, in turn, is based on our original two formulae:

Equation:

$$(x+a)^2 = x^2 + 2\text{ax} + a^2$$

Equation:

$$(x-a)^2 = x^2 - 2\text{ax} + a^2$$

As an example, consider the equation $x^2 + 10x + 21 = 0$. In order to make it fit one of the patterns above, we must replace the 21 with the correct number: a number such that $x^2 + 10x + \_\_$ is a perfect square. What number goes there? If you are familiar with the pattern, you know the answer right away. 10 is 5 **doubled**, so the number there must be 5 **squared**, or 25.

But how do we turn a 21 into a 25? We add 4, of course. And if we add 4 to **one** side of the equation, we have to add 4 to the **other** side of the equation. So the entire problem is worked out as follows:

Example:

Solving by Completing the Square (quick-and-dirty version)		
Solve $x^2 + 10x + 21 = 0$		**The problem.**
$x^2 + 10x + 25 = 4$		**Add 4 to both sides, so that the left side becomes a perfect square.**
$(x+5)^2 = 4$		**Rewrite the perfect square.**
$x + 5 = 2$ $x = -3$	$x + 5 = -2$ $x = -7$	**If something-squared is 4, the something can be 2, or -2 . Solve both possibilities to find the two answers.**

Thus, we have our two solutions.

Completing the square is more time-consuming than factoring: so whenever a quadratic equation can be factored, factoring is the preferred method. (In this case, we would have factored the original equation as $(x+3)(x+7)$ and gotten straight to the answer.) However, completing the square can be used on **any** quadratic equation. In the example below, completing the square is used to find two answers that would not have been found by factoring.

Example:

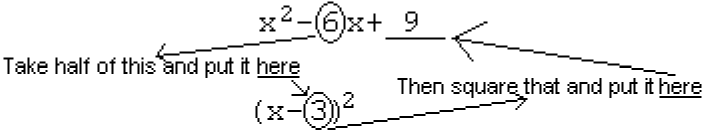
Solving by Completing the Square (showing all the steps more carefully)	
Solve $9x^2 - 54x + 80 = 0$	**The problem.**
$9x^2 - 54x = -80$	**Put all the x terms on one side, and the number on the other**
$x^2 - 6x = -\frac{80}{9}$	**Divide both sides by the coefficient of x^2 (*see below)**
$x^2 - 6x + 9 = -\frac{80}{9} + \underline{9}$	**Add the same number (*see below) to both sides, so that the left side becomes a perfect square.**
$(x-3)^2 = -\frac{80}{9} + \frac{81}{9} = \frac{1}{9}$	**Rewrite the perfect square.**

Solving by Completing the Square (showing all the steps more carefully)		
$x-3=\frac{1}{3}$ $x=3\frac{1}{3}$	$x-3=-\frac{1}{3}$ $x=2\frac{2}{3}$	**If something-squared is $\frac{1}{9}$, the something can be $\frac{1}{3}$, or $-\frac{1}{3}$. Solve both possibilities to find the two answers.**

Two steps in particular should be pointed out here.

In the third step, we divide both sides by 9. When completing the square, you do not want to have any coefficient in front of the term; if there is a number there, you divide it out. Fractions, on the other hand (such as the $-\frac{80}{9}$ in this case) do not present a problem. This is in marked contrast to factoring, where a coefficient in front of the x^2 can be left alone, but fractions make things nearly impossible.

The step after that is where we actually complete the square. $x^2+6\text{x}+\_\_$ will be our perfect square. How do we find what number we want? Start with the coefficient of x^2 (in this case, 6). **Take half of it, and square the result**. Half of 6 is 3, squared is 9. So we want a 9 there to create $x^2+6\text{x}+9$ which can be simplified to $(x+3)^2$.


x²-(6)x+ 9
Take half of this and put it here
Then square that and put it here
(x-(3))²

If the coefficient of x is an **odd number**, the problem becomes a little uglier, but the principle is the same. For instance, faced with:
Equation:

$$x^2+5\text{x}+\_\_$$

You would begin by taking half of 5 (which is $\frac{5}{2}$) and then squaring it:

$$x^2 + 5\text{x} + \frac{25}{4} = \left(x + \frac{5}{2}\right)^2$$

Another “completing the square” example, in which you cannot get rid of the square root at all, is presented in the worksheet “The Generic Quadratic Equation.”

One final note on completing the square: there are three different possible outcomes.

- If you end up with something like $\left(x - 3\right)^2 = 16$ you will find two solutions, since $x - 3$ can be either 4, or -4. You will always have two solutions if the right side of the equation, after completing the square, is positive.
- If you end up with $\left(x - 3\right)^2 = 0$ then there is only one solution: x must be 3 in this example. If the right side of the equation is 0 after completing the square, there is only one solution.
- Finally, what about a negative number on the right, such as $\left(x - 3\right)^2 = -16$? Nothing squared can give a negative answer, so there is no solution.

Quadratic Concepts -- The Quadratic Formula

This module discusses the quadratic formula.

In "Solving Quadratic Equations by Completing the Square" I talked about the common mathematical trick of solving a problem once, **using letters instead of numbers**, and then solving specific problems by plugging numbers into a general solution.

In the text, you go through this process for quadratic equations in general. The definition of a quadratic equation is any equation that can be written in the form:

Equation:

$$\mathrm{ax}^2+\mathrm{bx}+c=0$$

where $a\neq 0$. By completing the square on this generic equation, you arrive at the quadratic formula:

Equation:

$$x=\frac{-b\pm\sqrt{b^2-4\mathrm{ac}}}{2\mathrm{a}}$$

This formula can then be used to solve any quadratic equation, without having to complete the square each time. To see how this formula works, let us return to the previous problem:

Equation:

$$9\mathrm{x}^2-54x+80=0$$

In this case, $a=9$, $b=-54$, and $c=80$. So the quadratic formula tells us that the answers are:

Equation:

$$x=\frac{-(-54)\pm\sqrt{(-54)^2-4(9)(80)}}{2(9)}$$

We’ll use a calculator here rather than squaring 54 by hand....
Equation:

$$x = \frac{54 \pm \sqrt{2916 - 2880}}{18} = \frac{54 \pm \sqrt{36}}{18} = \frac{54 \pm 6}{18} = \frac{9 \pm 1}{3}$$

So we find that the two answers are $\frac{10}{3}$ and $\frac{8}{3}$, which are the same answers we got by completing the square.

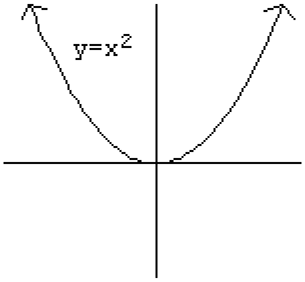
Using the quadratic formula is usually faster than completing the square, though still slower than factoring. So, in general, try to factor first: if you cannot factor, use the quadratic formula.

So why do we learn completing the square? Two reasons. First, completing the square is how you **derive** the quadratic formula. Second, completing the square is vital to **graphing** quadratic functions, as you will see a little further on in the chapter.

Quadratic Concepts -- Graphing Quadratic Equations

This module covers the graphing of quadratic equations.

The graph of the simplest quadratic function, $y = x^2$, looks like this:

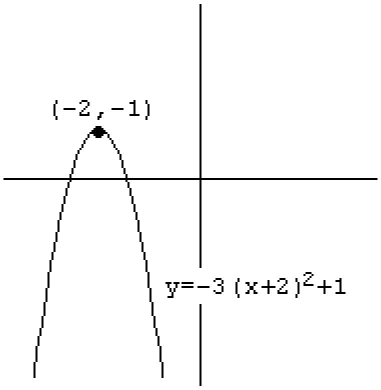


(You can confirm this by plotting points.) The point at the bottom of the U-shaped curve is known as the "vertex."

Now consider the function $y = -3(x+2)^2 + 1$. It's an intimidating function, but we have all the tools we need to graph it, based on the **permutations** we learned in the first unit. Let's step through them one by one.

- What does the – sign do? It multiplies all y-values by -1; positive values become negative, and vice-versa. So we are going to get an **upside-down** U-shape. We say that $y = x^2$ "opens up" and $y = -x^2$ "opens down."
- What does the 3 do? It multiplies all y-values by 3; positive values become more positive, and negative values become more negative. So it **vertical stretches** the function.
- What does the $+1$ at the end do? It adds 1 to all y-values, so it moves the function **up** by 1.
- Finally, what does the $+2$ do? This is a horizontal modification: if we plug in $x = 10$, we will be evaluating the function at $x = 12$. In general, we will always be copying the original x^2 function to our **right**; so we will be 2 units to the **left** of it.

So what does the graph look like? It has moved 2 to the left and 1 up, so the vertex moves from the origin $(0,0)$ to the point $(-2,1)$. The graph has also flipped upside-down, and stretched out vertically.



So graphing quadratic functions is easy, no matter how complex they are, if you understand permutations—and if the functions are written in the form $y = a(x - h)^2 + k$, as that one was.

Graphing Quadratic Functions

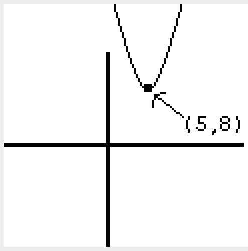
The graph of a quadratic function is always a vertical parabola. If the function is written in the form $y = a(x - h)^2 + k$ then the vertex is at (h,k). If a is positive, the parabola opens up; if a is negative, the parabola opens down.

But what if the functions are not expressed in that form? We’re more used to seeing them written as $y = \mathrm{ax}^2 + \mathrm{bx} + c$. For such a function, you graph it by first putting it into the form we used above, and then graphing it. And the way you get it into the right form is...completing the square! This process is almost identical to the way we used completing the square to **solve** quadratic equations, but some of the details are different.

Example:

Graphing a Quadratic Function

Graphing a Quadratic Function	
Graph $2x^2 - 20x + 58$	**The problem**.
$2(x^2 - 10x) + 58$	**We used to start out by dividing both sides by the coefficient of x^2 (2 in this case). In this case, we don't have another side: we can't make that 2 go away. But it's still in the way of completing the square. So we factor it out of the first two terms. Do not factor it out of the third (numerical) term; leave that part alone, outside of the parentheses.**
$2(x^2 - 10x + \underline{25}) + 58 - \underline{50}$	**Inside the parentheses, add the number you need to complete the square. (Half of 10, squared.)Now, when we add 25 inside the parentheses, what we have really done to our function? We have added 50, since everything in parentheses is doubled. So we keep the function the same by subtracting that 50 right back again, outside the parentheses! Since all we have done in this step is add 50 and then subtract it, the function is unchanged.**

Graphing a Quadratic Function	
$2(x-5)^2+8$	**Inside the parentheses, you now have a perfect square and can rewrite it as such. Outside the parentheses, you just have two numbers to combine.**
Vertex $(5{,}8)$ opens up	**Since the function is now in the correct form, we can read this information straight from the formula and graph it. Note that the number inside the parentheses (the h) always changes sign; the number outside (the k) does not.**
	**So there’s the graph! It’s easy to draw once you have the vertex and direction. It’s also worth knowing that the 2 vertically stretches the graph, so it will be thinner than a normal x^2.**

This process may look intimidating at first. For the moment, don’t worry about mastering the whole thing—instead, look over every **individual step** carefully and make sure you understand why it works—that is, why it keeps the function fundamentally unchanged, while moving us toward our goal of a form that we can graph.

The good news is, this process is basically the same every time. A different example is worked through in the worksheet “Graphing Quadratic

Functions II”—that example differs only because the x^2 term does not have a coefficient, which changes a few of the steps in a minor way. You will have plenty of opportunity to practice this process, which will help you get the “big picture” if you understand all the individual steps.

And don’t forget that what we’re really creating here is an algebraic generalization!

Equation:

$$2\mathrm{x}^2 - 20x + 58 = 2(x-5)^2 + 8$$

This is exactly the sort of generalization we discussed in the first unit—the assertion that these two very different functions will **always give the same answer** for any x-value you plug into them. For this very reason, we can also assert that the two graphs will look the same. So we can graph the first function by graphing the second.

Quadratic Concepts -- Solving Problems by Graphing Quadratic Functions

Surprisingly, there is a fairly substantial class of real world problems that can be solved by graphing quadratic functions.

These problems are commonly known as “optimization problems” because they involve the question: “When does this important function reach its **maximum**?” (Or sometimes, its **minimum**?) In real life, of course, there are many things we want to maximize—a company wants to maximize its revenue, a baseball player his batting average, a car designer the leg room in front of the driver. And there are many things we want to minimize—a company wants to minimize its costs, a baseball player his errors, a car designer the amount of gas used. Mathematically, this is done by writing a function for that quantity and finding where that function reaches its highest or lowest point.

Example:
Exercise:

Problem:

If a company manufactures x items, its total cost to produce these items is $x^3 - 10x^2 + 43x$. How many items should the company make in order to **minimize** its **average cost per item**?

Solution:

“Average cost per item” is the total cost, divided by the number of items. For instance, if it costs $600 to manufacture 50 items, then the average cost per item was $12. It is important for companies to minimize average cost because this enables them to sell at a low price.

In this case, the total cost is $x^3 - 10x^2 + 43x$ and the number of items is x. So the **average cost per item** is

Equation:

$$\mathrm{A}(x) = \frac{x^3 - 10x^2 + 43x}{x} = x^2 - 10x + 43$$

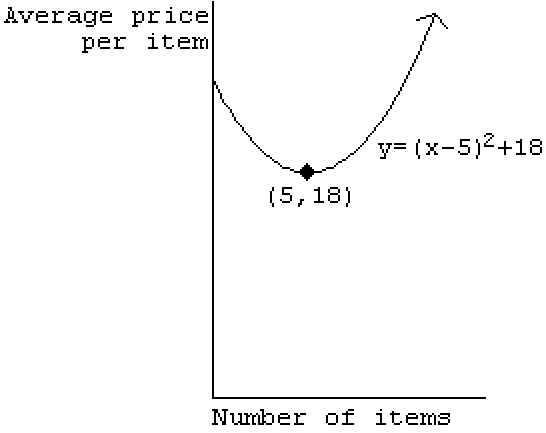
What the question is asking, mathematically, is: what value of x makes this function the **lowest**?

Well, suppose we were to graph this function. We would complete the square by rewriting it as:

Equation:

$$A(x) = x^2 - 10x + 25 + 18 = (x-5)^2 + 18$$

The graph opens up (since the $(x-5)^2$term is positive), and has its vertex at $(5{,}18)$ (since it is moved 5 to the right and 18 up). So it would look something like this:



I have graphed only the first quadrant, because negative values are not relevant for this problem (why?).

The real question here is, what can we learn from that graph? Every point on that graph represents one possibility for our company: if they manufacture x items, the graph shows what $A(x)$, the average cost per item, will be.

The point $(5{,}18)$ is the **lowest point on the graph**. It is possible for x to be higher or lower than 5, but it is never possible for A to be lower than 18. So if its goal is to minimize average cost, their best strategy is to manufacture 5 items, which will bring their average cost to $18/item.

Rational Expression Concepts -- Introduction

This module provides an introduction to rational expressions.

The term “rational” in math is not used in the sense of “sane” or “sensible.” It is instead used to imply a **ratio**, or fraction. A **rational expression** is the ratio of two polynomials: for instance, $\frac{x^2+1}{x^2-1}$ is a rational expression.

There are two rules for working with rational expressions.

1. Begin every problem by factoring everything you can.
2. Remember that, despite all the complicated looking functions, a rational expression is just a fraction: you manipulate them using all the rules of fractions that you are familiar with.

Rational Expression Concepts -- Simplifying Rational Expressions

This module provides techniques for simplifying rational expressions.

How do you **simplify** a fraction? The answer is, you divide the top and bottom by the same thing.

Equation:

$$\frac{4}{6} = \frac{4 \div 2}{6 \div 2} = \frac{2}{3}$$

So $\frac{4}{6}$ and $\frac{2}{3}$ are two different ways of writing the same number.

On the left, a pizza divided into six equal slices: the four shaded-in regions represent $\frac{4}{6}$ of a pizza. On the right, a pizza divided into three equal slices: the two shaded-in regions represent $\frac{2}{3}$ of a pizza. The two areas are identical: $\frac{4}{6}$ and $\frac{2}{3}$ are two different ways of expressing the **same amount of pizza**.

In some cases, you have to repeat this process more than once before the fraction is fully simplified.

Equation:

$$\frac{40}{48} = \frac{40 \div 4}{48 \div 4} = \frac{10}{12} = \frac{10 \div 2}{12 \div 2} = \frac{5}{6}$$

It is vital to remember that **we have not divided this fraction by 4, or by 2, or by 8**. We have rewritten the fraction in another form: $\frac{40}{48}$ is the **same number** as $\frac{5}{6}$. In strictly practical terms, if you are given the choice between $\frac{40}{48}$ of a pizza or $\frac{5}{6}$ of a pizza, it does not matter which one you choose, because they are the same amount of pizza.

You can divide the top and bottom of a fraction by the same number, but you cannot subtract the same number from the top and bottom of a fraction!

$\frac{40}{48} = \frac{40-39}{48-39} = \frac{1}{9}$ ✗ Wrong!

Given the choice, a hungry person would be wise to choose $\frac{40}{48}$ of a pizza instead of $\frac{1}{9}$.

Dividing the top and bottom of a fraction by the same number leaves the fraction unchanged, and that is how you simplify fractions. **Subtracting** the same number from the top and bottom changes the value of the fraction, and is therefore an illegal simplification.

All this is review. But if you understand these basic fraction concepts, you are ahead of many Algebra II students! And if you can **apply these same concepts when variables are involved**, then you are ready to simplify rational expressions, because there are no new concepts involved.

As an example, consider the following:
Equation:

$$\frac{x^2 - 9}{x^2 + 6\text{x} + 9}$$

You might at first be tempted to cancel the common x^2 terms on the top and bottom. But this would be, mathematically, **subtracting** x^2 from both the top and the bottom; which, as we have seen, is an illegal fraction operation.

$\frac{x^2-9}{x^2+6\text{x}+9}$	$=\frac{\cancel{x^2}-9}{\cancel{x^2}+6x+9}$	$=\frac{-9}{6\text{x}+9}$	**✗ Wrong!**

To properly simplify this expression, begin by factoring both the top and the bottom, and **then** see if anything cancels.

Example:
Simplifying Rational Expressions

$\frac{x^2-9}{x^2+6\text{x}+9}$	**The problem**
$=\frac{(x+3)(x-3)}{(x+3)^2}$	**Always begin rational expression problems by factoring! This factors easily, thanks to** $(x+a)(x-a)=x^2-a^x$ and $(x+a)^2=x^2+2\text{ax}+a^2$
$=\frac{x-3}{x+3}$	**Cancel a common** $(x+3)$ **term on both the top and the bottom. This is legal because this term was multiplied on both top and bottom; so we are effectively dividing the top and bottom by** $(x+3)$**, which leaves the fraction unchanged**.

What we have created, of course, is an algebraic generalization:
Equation:

$$\frac{x^2 - 9}{x^2 + 6x + 9} = \frac{x - 3}{x + 3}$$

For any x value, the complicated expression on the left will give the same answer as the much simpler expression on the right. You may want to try one or two values, just to confirm that it works.

As you can see, the skills of **factoring** and **simplifying fractions** come together in this exercise. No new skills are required.

Rational Expression Concepts -- Multiplying Rational Expressions

This module covers the multiplication of rational expressions.

Multiplying fractions is easy: you just multiply the tops, and multiply the bottoms. For instance,

Equation:

$$\frac{6}{7}\times\frac{7}{11}=\frac{6\times 7}{7\times 11}=\frac{42}{77}$$

Now, you may notice that $\frac{42}{77}$ can be simplified, since 7 goes into the top and bottom. $\frac{42}{77}=\frac{42\div 7}{77\div 7}=\frac{6}{11}$. So $\frac{42}{77}$ is the correct answer, but $\frac{6}{11}$ is **also** the correct answer (since they are the same number), and it’s a good bit simpler.

In fact, we could have jumped straight to the simplest answer first, and avoided dealing with all those big numbers, if we had noticed that we have a 7 in the numerator and a 7 in the denominator, and **cancelled** them before we even multiplied!

$$\frac{6}{\cancel{7}}\ x\ \frac{\cancel{7}}{11}=\frac{6}{11}$$

This is a great time-saver, and you’re also a lot less likely to make mistakes.

When multiplying fractions...

If the same number appears on the top and the bottom, you can cancel it before you multiply. This works regardless of whether the numbers appear in the **same fraction or different fractions**.

But it’s critical to remember that this rule **only applies when you are multiplying fractions**: not when you are adding, subtracting, or dividing.

As you might guess, all this review of basic fractions is useful because, once again, rational expressions work the same way.

Example: **Multiplying Rational Expressions**	
$\frac{3x^2-21x-24}{x^2-16} \cdot \frac{x^2-6x+8}{3x+3}$	**The problem**
$= \frac{3(x-8)(x+1)}{(x+4)(x-4)} \cdot \frac{(x-2)(x-4)}{3(x+1)}$	**Always begin rational expression problems by factoring! Note that for the first element you begin by factoring out the common 3, and then factoring the remaining expression.**
$= \frac{\cancel{3}(x-8)\cancel{(x+1)}}{(x+4)\cancel{(x-4)}} \cdot \frac{(x-2)\cancel{(x-4)}}{\cancel{3}\cancel{(x+1)}}$	**When multiplying fractions, you can cancel anything on top with anything on the bottom, even across different fractions**
$= \frac{(x-8)(x-2)}{x+4}$	**Now, just see what you’re left with. Note that you could rewrite the top as $x^2-10x+16$ but it’s generally easier to work with in factored form.**

Dividing Rational Expressions

To divide fractions, you flip the bottom one, and then multiply.
Equation:

$$\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \cdot 3 = \frac{3}{2}$$

After the “flipping” stage, all the considerations are exactly the same as multiplying.

Example:
Dividing Rational Expressions

$\frac{\frac{x^2-3x}{2x^2-13x+6}}{\frac{x^3+4x}{x^2-12x+36}}$	This problem could also be written as: $\frac{x^2-3x}{2x^2-13x+6} \div \frac{x^3+4x}{x^2-12x+36}$. However, the symbol is rarely seen at this level of math. $12 \div 4$ is written as $\frac{12}{4}$.
$\frac{x^2-3x}{2x^2-13x+6} \times \frac{x^2-12x+36}{x^3+4x}$	Flip the bottom and multiply. From here, it’s a straight multiplication problem.
$= \frac{x(x-3)}{(2x-1)(x-6)} \times \frac{(x-6)^2}{x(x^2+4)}$	Always begin rational expression problems by factoring! Now, cancel a factor of x and an $(x-6)$ and you get...
$= \frac{(x-3)(x-6)}{(2x-1)(x^2+4)}$	That’s as simple as it gets, I’m afraid. But it’s better than what we started with!

Rational Expression Concepts -- Adding and Subtracting Rational Expressions

This module covers the addition and subtraction of rational expressions.

Adding and subtracting fractions is harder—but once again, it is a familiar process.

Equation:

$$\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$$

The key is finding the **least common denominator**: the smallest multiple of both denominators. Then you rewrite the two fractions with this denominator. Finally, you add the fractions by **adding the numerators and leaving the denominator alone**.

But how do you find the least common denominator? Consider this problem:

Equation:

$$\frac{5}{12}+\frac{7}{30}$$

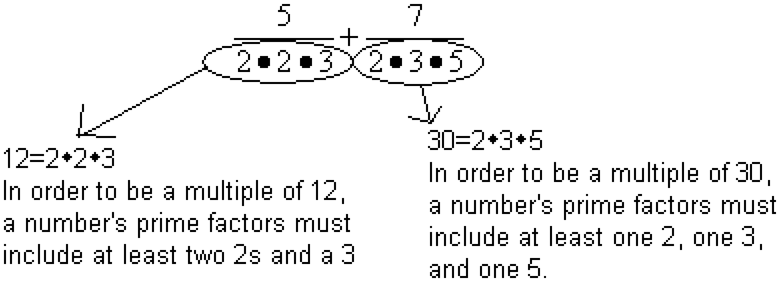
You could probably find the least common denominator if you played around with the numbers long enough. But what I want to show you is a **systematic method** for finding least common denominators—a method that works with rational expressions just as well as it does with numbers. We start, as usual, by factoring. For each of the denominators, we find all the **prime factors**, the prime numbers that multiply to give that number.

Equation:

$$\frac{5}{2\cdot 2\cdot 3}+\frac{7}{2\cdot 3\cdot 5}$$

If you are not familiar with the concept of prime factors, it may take a few minutes to get used to. $2\times 2\times 3$ is 12, broken into its **prime factors**: that is, it is the list of prime numbers that multiply to give 12. Similarly, the prime factors of 30 are $2 \times 3 \times 5$.

Why does that help? Because $12 = 2 \times 2 \times 3$, any number whose prime factors include two 2s and one 3 will be a multiple of 12. Similarly, any number whose prime factors include a 2, a 3, and a 5 will be a multiple of 30.



12=2•2•3
In order to be a multiple of 12, a number's prime factors must include at least two 2s and a 3

30=2•3•5
In order to be a multiple of 30, a number's prime factors must include at least one 2, one 3, and one 5.

The least common denominator is the smallest number that meets both these criteria: it must have two 2s, one 3, and one 5. Hence, the least common denominator must be $2 \times 2 \times 3 \times 5$, and we can finish the problem like this.

Equation:

$$\frac{5}{2\cdot 2\cdot 3}+\frac{7}{2\cdot 3\cdot 5}=\frac{5\underline{5}}{(2\cdot 2\cdot 3)\underline{5}}+\frac{7\underline{2}}{(2\cdot 3\cdot 5)\underline{2}}=\frac{25}{60}+\frac{14}{60}=\frac{39}{60}$$

This may look like a very strange way of solving problems that you’ve known how to solve since the third grade. However, I would urge you to spend a few minutes carefully following that solution, focusing on the question: **why is $2 \times 2 \times 3 \times 5$ guaranteed to be the least common denominator**? Because once you understand that, you have the key concept required to add and subtract rational expressions.

Example:
Subtracting Rational Expressions

$\frac{3}{x^2+12x+36} - \frac{4x}{x^3+4x^2-12x}$	**The problem**
$= \frac{3}{(x+6)^2} - \frac{4x}{x(x+6)(x-2)}$	**Always begin rational expression problems by factoring! The least common denominator must have two $(x+6)$ s, one x , and one $(x-2)$.**
$= \frac{3(x)(x-2)}{(x+6)^2(x)(x-2)} - \frac{4x(x+6)}{x(x+6)^2(x-2)}$	**Rewrite both fractions with the common denominator.**
$= \frac{3(x)(x-2)-4x(x+6)}{x(x-2)(x+6)^2}$	**Subtracting fractions is easy when you have a common denominator! It’s best to leave the bottom alone, since it is factored. The top, however, consists of two separate factored pieces, and will be simpler if we multiply them out so we can combine them.**
$= \frac{3x^2-6x-(4x^2+24x)}{x(x-2)(x+6)^2}$	**A common student mistake here is forgetting the parentheses. The entire second term is subtracted; without the parentheses, the $24x$ ends up being added.**
$= \frac{-x^2-30x}{x(x-2)(x+6)^2}$	**Almost done! But finally, we note that we can factor the top again. If we factor out an x it will cancel with the x in the denominator.**

$= \frac{-x-30}{(x-2)(x+6)^2}$	**A lot simpler than where we started, isn’t it?**

The problem is long, and the math is complicated. So after following all the steps, it’s worth stepping back to realize that even this problem results simply from the two rules we started with.

First, always factor rational expressions before doing anything else.

Second, follow the regular processes for fractions: in this case, the procedure for subtracting fractions, which involves finding a common denominator. After that, you subtract the numerators while leaving the denominator alone, and then simplify.

Rational Expression Concepts -- Rational Equations
This module introduces rational expressions in equations.

Rational Equations

A **rational equation** means that you are setting two rational **expressions** equal to each other. The goal is to solve for x; that is, find the x value(s) that make the equation true.

Suppose I told you that:
Equation:

$$\frac{x}{8}=\frac{3}{8}$$

If you think about it, the x in this equation has to be a 3. That is to say, if x=3 then this equation is true; for any other x value, this equation is false.

This leads us to a very general rule.

A very general rule about rational equations

If you have a rational equation where the **denominators** are the same, then the **numerators** must be the same.

This in turn suggests a strategy: find a common denominator, and then set the numerators equal.

Example: Rational Equation

Example: Rational Equation	
$\frac{3}{x^2+12x+36} = \frac{4x}{x^3+4x^2-12x}$	**Same problem we worked before, but now we are equating these two fractions, instead of subtracting them.**
$\frac{3(x)(x-2)}{(x+6)^2(x)(x-2)} = \frac{4x(x+6)}{x(x+6)^2(x-2)}$	**Rewrite both fractions with the common denominator.**
$3x(x-2) = 4x(x+6)$	**Based on the rule above—since the denominators are equal, we can now assume the numerators are equal.**
$3x^2-6x = 4x^2+24x$	**Multiply it out**
$x^2+30x=0$	**What we're dealing with, in this case, is a quadratic equation. As always, move everything to one side...**
$x(x+30)=0$	**...and then factor. A common mistake in this kind of problem is to divide both sides by x; this loses one of the two solutions.**
$x=0$ or $x=-30$	**Two solutions to the quadratic equation. However, in this case, $x=0$ is not valid, since it was not in the domain of the original right-hand fraction. (Why?) So this problem actually has only one solution, $x=-30$.**

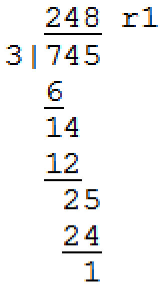
As always, it is vital to remember what we have found here. We started with the equation $\frac{3}{x^2+12x+36} = \frac{4\text{x}}{x^3+4\text{x}^2-12x}$. We have concluded now that if you plug $x = -30$ into that equation, you will get a true equation (you can verify this on your calculator). For any other value, this equation will evaluate false.

To put it another way: if you graphed the functions $\frac{3}{x^2+12x+36}$ and $\frac{4\text{x}}{x^3+4\text{x}^2-12x}$, the two graphs would intersect at one point only: the point when $x = -30$.

Rational Expression Concepts -- Dividing Polynomials

Simplifying, multiplying, dividing, adding, and subtracting rational expressions are all based on the basic skills of working with fractions. Dividing polynomials is based on an even earlier skill, one that pretty much everyone remembers with horror: long division.

To refresh your memory, try dividing $\frac{745}{3}$ by hand. You should end up with something that looks something like this:

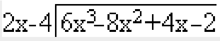
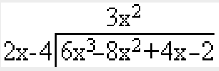
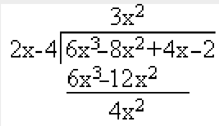
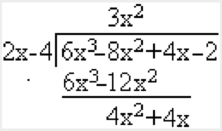
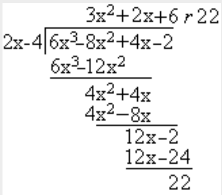


```
   248 r1
 3|745
   6
   14
   12
    25
    24
     1
```

So we conclude that $\frac{745}{3}$ is 248 with a remainder of 1; or, to put it another way, $\frac{745}{3} = 248\frac{1}{3}$.

You may have decided years ago that you could forget this skill, since calculators will do it for you. But now it comes roaring back, because here is a problem that your calculator will not solve for you: $\frac{6x^3-8x^2+4x-2}{2x-4}$. You can solve this problem in much the same way as the previous problem.

Example:	
$\frac{6x^3-8x^2+4x-2}{2x-4}$	**The problem**

 	**The problem, written in standard long division form.**
 	**Why $3x^2$? This comes from the question: “How many times does $2x$ go into $6x^3$?” Or, to put the same question another way: “What would I multiply $2x$ by, in order to get $6x^3$?” This is comparable to the first step in our long division problem: “What do I multiply 3 by, to get 7?”**
 	**Now, multiply the $3x^2$ times the $(2x - 4)$ and you get $6x^3 - 12x^2$. Then subtract this from the line above it. The $6x^3$ terms cancel—that shows we picked the right term above! Note that you have to be careful with signs here. $-8x^2 - \left(-12x^2\right)$ gives us positive $4x^2$.**
 	**Bring down the $4x$. We have now gone through all four steps of long division—divide, multiply, subtract, and bring down. At this point, the process begins again, with the question “How many times does $2x$ go into $4x^2$?”**
 3x²+2x+6 r 22 2x-4 | 6x³-8x²+4x-2 6x³-12x² 4x²+4x 4x²−8x 12x−2 12x−24 22	**This is not the next step...this is what the process looks like after you’ve finished all the steps. You should try going through it yourself to make sure it ends up like this.**

Polynomial Division

So we conclude that $\frac{6x^3 - 8x^2 + 4x - 2}{2x - 4}$ is $3x^2 + 2x + 6$ with a remainder of 22, or, to put it another way, $3x^2 + 2x + 6 + \frac{22}{2x - 4}$.

Checking your answers

As always, checking your answers is not just a matter of catching careless errors: it is a way of making sure that you **know what you have come up with**. There are two different ways to check the answer to a division problem, and both provide valuable insight

The first is by plugging in numbers. We have created an algebraic generalization:

Equation:

$$\frac{6\mathrm{x}^3 - 8\mathrm{x}^2 + 4\mathrm{x} - 2}{2\mathrm{x} - 4} = 3\mathrm{x}^2 + 2\mathrm{x} + 6 + \frac{22}{2\mathrm{x} - 4}$$

In order to be valid, this generalization must hold for $x = 3$, $x = -4$, $x = 0$, $x = \varpi$,or any other value except $x = 2$ (which is outside the domain). Let’s try $x = 3$.

Checking the answer by plugging in $x = 3$

Equation:

$$\frac{6(3)^3 - 8(3)^2 + 4(3) - 2}{2(3) - 4} \overset{?}{=} 3(3)^2 + 2(3) + 6 + \frac{22}{2(3) - 4}$$

Equation:

$$\frac{162 - 72 + 12 - 2}{6 - 4} \overset{?}{=} 27 + 6 + 6 + \frac{22}{6 - 4}$$

Equation:

$$\frac{100}{2} \overset{?}{=} 39 + \frac{22}{2}$$

Equation:

$$50 \overset{?}{=} 39 + 11\checkmark$$

The second method is by multiplying back. Remember what division is: it is the opposite of multiplication! If $\frac{745}{3}$ is 248 with a remainder of 1, that means that $248 \cdot 3$ will be 745, with 1 left over. Similarly, if our long division was correct, then $\left(3\mathrm{x}^2 + 2\mathrm{x} + 6\right)(2\mathrm{x} - 4) + 22$ should be $6\mathrm{x}^3 - 8\mathrm{x}^2 + 4\mathrm{x} - 2$.

Checking the answer by multiplying back

Equation:

$$\left(3\mathrm{x}^2 + 2\mathrm{x} + 6\right)(2\mathrm{x} - 4) + 22$$

Equation:

$$= \left(6\mathrm{x}^3 - 12x^2 + 4\mathrm{x}^2 - 8\mathrm{x} - 24\right) + 22$$

Equation:

$$= 6\mathrm{x}^3 - 8\mathrm{x}^2 + 4\mathrm{x} - 2\checkmark$$

Radical Concepts -- Introduction

This module provides an introduction to radical numbers.

The concept of a radical (or root) is a familiar one, and was reviewed in the conceptual explanation of logarithms in the previous chapter. In this chapter, we are going to explore some possibly unfamiliar properties of radicals, and solve equations involving radicals.

Radical Concepts -- Properties of Radicals

What is $\sqrt{x^2+9}$? Many students will answer quickly that the answer is $(x+3)$ and have a very difficult time believing this answer is wrong. But it is wrong.

$\sqrt{x^2}$ **is** x^* [footnote] and $\sqrt{9}$ **is** 3, but $\sqrt{x^2+9}$ **is not** $(x+3)$.
I’m fudging a bit here: $\sqrt{x^2}$ is x only if you ignore negative numbers. For instance, if $x = -3$, then $x^2 = 9$, and $\sqrt{x^2}$ is 3; so in that case, $\sqrt{x^2}$ is not x. In general, $\sqrt{x^2} = |x|$. However, this subtlety is not relevant to the overall point, which is that you cannot break up two terms that are added under a radical.

Why not? Remember that $\sqrt{x^2+9}$ is asking a question: “what **squared** gives the answer x^2+9 ?” So $(x+3)$ is not an answer, because $(x+3)^2 = x^2 +$ 6x+9, **not** x^2+9 .

As an example, suppose $x = 4$. So $\sqrt{x^2+9} = \sqrt{4^2+9} = \sqrt{25} = 5$. But $(x+3) = 7$.

Note: If two numbers are **added** or **subtracted** under a square root, you cannot split them up. In symbols: $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ or, to put it another way, $\sqrt{x^2+y^2} \neq a+b$

$\sqrt{x^2+9}$ cannot, in fact, be simplified at all. It is a perfectly valid function, but cannot be rewritten in a simpler form.

How about $\sqrt{9x^2}$? By analogy to the previous discussion, you might expect that this cannot be simplified either. But in fact, it can be simplified:

$\sqrt{9x^2} = 3x$

Why? Again, $\sqrt{9x^2}$ is asking “what **squared** gives the answer $9x^2$?” The answer is $3x$ because $(3x)^2 = 9x^2$.

Similarly, $\sqrt{\frac{9}{x^2}} = \frac{3}{x}$, because $\left(\frac{3}{x}\right)^2 = \frac{9}{x^2}$.

Note: If two numbers are multiplied or divided under a square root, you **can** split them up. In symbols: $\sqrt{\text{ab}} = \sqrt{a}\sqrt{b}$, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Radical Concepts -- Simplifying Radicals
This module covers techniques for the simplification of radicals.

Simplifying Radicals

The property $\sqrt{\text{ab}}$= $\sqrt{a}\sqrt{b}$ can be used to simplify radicals. The key is to break the number inside the root into two factors, **one of which is a perfect square**.

Example:
Simplifying a Radical

$\sqrt{75}$	
$= \sqrt{25 \bullet 3}$	**because 25•3 is 75, and 25 is a perfect square**
$= \sqrt{25}\sqrt{3}$	**because** $\sqrt{\text{ab}} = \sqrt{a}\ \sqrt{b}$
$= 5\ \sqrt{3}$	**because** $\sqrt{25}$ **=5**

So we conclude that $\sqrt{75}$=5 $\sqrt{3}$. You can confirm this on your calculator (both are approximately 8.66).

We rewrote 75 as $25 \bullet 3$ because 25 is a perfect square. We could, of course, also rewrite 75 as $5 \bullet 15$, but—although correct—that would not help us simplify, because neither number is a perfect square.

Example:
Simplifying a Radical in Two Steps

$\sqrt{180}$	
$= \sqrt{9 \bullet 20}$	**because** $9 \bullet 20$ **is 180, and 9 is a perfect square**
$= \sqrt{9}\,\sqrt{20}$	**because** $\sqrt{\mathrm{ab}} = \sqrt{a}\,\sqrt{b}$
$= 3\,\sqrt{20}$	**So far, so good. But wait! We're not done!**
$= 3\,\sqrt{4 \bullet 5}$	**There's another perfect square to pull out!**
$= 3\,\sqrt{4}\sqrt{5}$	
$= 3(2)\,\sqrt{5}$	
$= 6\,\sqrt{5}$	**Now we're done.**

The moral of this second example is that **after** you simplify, you should always look to see if you can simplify **again** .

A secondary moral is, try to pull out the biggest perfect square you can. We could have jumped straight to the answer if we had begun by rewriting 180 as $36 \bullet 5$.

This sort of simplification can sometimes allow you to **combine** radical terms, as in this example:

Example:
Combining Radicals

$\sqrt{75}$ – $\sqrt{12}$	
$= 5\sqrt{3} – 2\sqrt{3}$	**We found earlier that $\sqrt{75} = 5\sqrt{3}$. Use the same method to confirm that $\sqrt{12} = 2\sqrt{3}$.**
$= 3\sqrt{3}$	**5 of anything minus 2 of that same thing is 3 of it, right?**

That last step may take a bit of thought. It can only be used when the radical is the same. Hence, $\sqrt{2}$ + $\sqrt{3}$ cannot be simplified at all. We were able to simplify $\sqrt{75}$ – $\sqrt{12}$ only by making the radical in both cases the same .

So why does $5\sqrt{3}–2\sqrt{3}= 3\sqrt{3}$? It may be simplest to think about verbally: 5 of these things, minus 2 of the same things, is 3 of them. But you can look at it more formally as a factoring problem, if you see a common factor of $\sqrt{3}$.

$$5\sqrt{3}–2\sqrt{3} = \sqrt{3}(5–2) = \sqrt{3}(3).$$

Of course, the process is exactly the same if variable are involved instead of just numbers!

Example:
Combining Radicals with Variables

$x^{\frac{3}{2}} + x^{\frac{5}{2}}$	
$= x^3 + x^5$	**Remember the definition of fractional exponents!**
$= \sqrt{x^{2} * x} + \sqrt{x^{4} * x}$	**As always, we simplify radicals by factoring them inside the root...**
$\sqrt{x^2} * \sqrt{x} + \sqrt{x^4} * \sqrt{x}$	**and then breaking them up...**
$= x\sqrt{x} + x^2\sqrt{x}$	**and then taking square roots outside!**
$= \left(x^2 + x\right)\sqrt{x}$	**Now that the radical is the same, we can combine.**

Rationalizing the Denominator

It is always possible to express a fraction with no square roots in the denominator.

Is it always desirable? Some texts are religious about this point: “You should never have a square root in the denominator.” I have absolutely no idea why. To me, $\frac{1}{\sqrt{2}}$ looks simpler than $\frac{\sqrt{2}}{2}$; I see no overwhelming reason for forbidding the first or preferring the second.

However, there are times when it is useful to remove the radicals from the denominator: for instance, when adding fractions. The trick for doing this is based on the basic rule of fractions: **if you multiply the top and bottom of a fraction by the same number, the fraction is unchanged.** This rule enables us to say, for instance, that $\frac{2}{3}$ is exactly the same number as $\frac{2 \cdot 3}{3 \cdot 3} = \frac{6}{9}$
.

In a case like $\frac{1}{\sqrt{2}}$, therefore, you can multiply the top and bottom by $\sqrt{2}$.

$$\frac{1}{\sqrt{2}} = \frac{1*2}{\sqrt{2}*\sqrt{2}} = \frac{\sqrt{2}}{2}$$

What about a more complicated case, such as $\frac{\sqrt{12}}{1+\sqrt{3}}$? You might think we could simplify this by multiplying the top and bottom by $(1+\sqrt{3})$, but that doesn't work: the bottom turns into $(1+3)^2 = 1 + 2\sqrt{3}+3$, which is at least as ugly as what we had before.

The correct trick for getting rid of $(1+\sqrt{3})$ is to multiply it by $(1-\sqrt{3})$. These two expressions, identical except for the replacement of $a+$ by a-, are known as **conjugates**. What happens when we multiply them? We don't need to use FOIL if we remember that

$$\left(x+y\right)\left(x-y\right) = x^2 - y^2$$

Using this formula, we see that

$$\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\right) = 1^2 - \left(\sqrt{3}\right)^2 = 1 - 3 = -2$$

So the square root does indeed go away. We can use this to simplify the original expression as follows.

Example:
Rationalizing Using the Conjugate of the Denominator

$$\frac{\sqrt{12}}{1+\sqrt{3}} = \frac{\sqrt{12}\left(1-\sqrt{3}\right)}{\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\right)} = \frac{\sqrt{12}-\sqrt{36}}{1-3} = \frac{2\sqrt{3}-6}{-2} = -\sqrt{3}+3$$

As always, you may want to check this on your calculator. Both the original and the simplified expression are approximately 1.268.

Of course, the process is the same when variables are involved.

Example:
Rationalizing with Variables

$$\frac{1}{x-\sqrt{x}} = \frac{1\left(x+\sqrt{x}\right)}{\left(x-\sqrt{x}\right)\left(x+\sqrt{x}\right)} = \frac{x+\sqrt{x}}{x^2-x}$$

Once again, we multiplied the top and the bottom by the **conjugate of the denominator**: that is, we replaced a- with $a+$. The formula $\left(x+a\right)\left(x-a\right) = x^2 - a^2$ enabled us to quickly multiply the terms on the bottom, and eliminated the square roots in the denominator.

Radical Concepts -- Radical Equations
This module contains methods on solving radical equations.

When solving **equations that involve radicals**, begin by asking yourself: **is there an under the square root?** The answer to this question will determine the way you approach the problem.

If there is **not** an under the square root—if only numbers are under the radicals—you can solve much the same way you would solve with no radicals at all.

Example:
Radical Equation with No Variables Under Square Roots

– –	Sample problem: no variables under radicals
– –	Get everything with an on one side, everything else on the other
– –	Factor out the
– –	Divide, to solve for

The key thing to note about such problems is that **you do not have to square both sides of the equation**. ‾ may look ugly, but it is just a number—you could find it on your calculator if you wanted to—it functions in the equation just the way that the number 10, or –, or π would.

If there is an under the square root, the problem is completely different. You will have to square both sides to get rid of the radical. However, there are two important notes about this kind of problem.

1. Always get the radical **alone, on one side of the equation**, before squaring.
2. Squaring both sides can introduce **false answers**—so it is important to check your answers after solving!

Both of these principles are demonstrated in the following example.

Example:
Radical Equation with Variables under Square Roots

	Sample problem with variables under radicals
	Isolate the radical before squaring!
	Now, square both sides
	Multiply out. Hey, it looks like a quadratic equation now!
	As always with quadratics, get everything on one side.
	Factoring: the easiest way to solve quadratic equations.
− or	Two solutions**. Do they work?** Check in the **original equation**!

Check −	**Check**

– – –	
– – – –	–
– – – –	
– – – –	**Not equal!**
– –	

So the algebra yielded two solutions: – and –1. Checking, however, we discover that only the first solution is valid. This problem demonstrates how important it is to check solutions whenever squaring both sides of an equation.

If variables under the radical occur more than once, you will have to go through this procedure multiple times. Each time, you isolate a radical and then square both sides.

Example: **Radical Equation with Variables under Square Roots Multiple Times**	
	Sample problem with variables under radicals multiple times
–	Isolate one radical. (I usually prefer to start with the bigger one.)

	Square both sides. The **two**-radical equation is now a **one**-radical equation.
	Isolate the remaining radical, then square both sides again..
	In this case, we end up with only one solution. But we still need to check it.
Check x=9	

Remember, the key to this problem was recognizing that **variables under the radical** occurred in the original problem **two times**. That cued us that we would have to go through the process—isolate a radical, then square both sides—twice, before we could solve for . And **whenever** you square both sides of the equation, it’s vital to check your answer(s)!

When good math leads to bad answers

Why is it that—when squaring both sides of an equation—perfectly good algebra can lead to invalid solutions? The answer is in the redundancy of squaring. Consider the following equation:

False. But square both sides, and we get...

25 = 25 True. So squaring both sides of a **false** equation can produce a **true** equation.

To see how this affects our equations, try plugging into the various steps of the first example.

Example:
Why did we get a false answer of x=–1 in Example 1?

	Does work here? No, it does not.
	How about here? No, produces the false equation 1=–1.
	Suddenly, works. (Try it!)

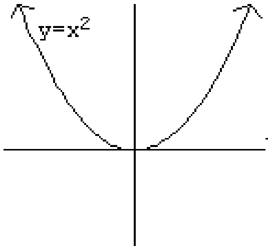
When we squared both sides, we “lost” the difference between 1 and –1, and they “became equal.” From here on, when we solved, we ended up with as a valid solution.

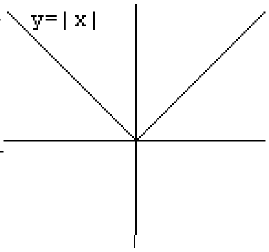
Test your memory: When you square both sides of an equation, you can introduce false answers. We have encountered one other situation where **good algebra** can lead to a **bad answer**. When was it?

Answer: It was during the study of absolute value equations, such as . In those equations, we also found the hard-and-fast rule that you **must check your answers** as the last step.

What do these two types of problem have in common? The function actually has a lot in common with . Both of them have the peculiar property that they always turn and into the same response. (For instance, if you plug –3 and 3 into the function, you get the same thing back.) This property is known as being an **even function**. Dealing with such “redundant” functions leads, in both cases, to the possibility of false answers.

The similarity between these two functions can also be seen in the graphs: although certainly not identical, they bear a striking resemblance to each other. In particular, both graphs are symmetric about the y-axis, which is the fingerprint of an “even function”.





Rational Expressions Homework -- Homework: Rational Expressions and Equations

Exercise:

Problem:

- **a**Simplify
- **b**What values of are not allowed in the original expression?
- **c**What values of are not allowed in your simplified expression?

Exercise:

Problem:

- **a**Simplify
- **b**What values of are not allowed in the original expression?
- **c**What values of are not allowed in your simplified expression?

Exercise:

Problem:

- **a**Simplify
- **b**What values of are not allowed in the original expression?
- **c**What values of are not allowed in your simplified expression?
- **d**Test your answer by choosing an value and plugging it into the original expression, and your simplified expression. Do they yield the same answer?

Exercise:

Problem: **<something>**

- **a**What is the something?
- **b**What values of are not allowed in the original expression?
- **c**What values of are not allowed in your simplified expression?

Exercise:

Problem: —— ——

- **a**Solve for . You should get two answers.
- **b**Check by plugging one of your answers back into the original equation.

Exercise:

Problem: If —, find ————.

Note: The first item, , is a composite function!

Simplify as much as possible.

Imaginary Concepts -- Introduction

This module introduces i, the imaginary number, and the concept of complex numbers in Algebra.

$$(\text{-}1)^2 = 1$$

$$1^2 = 1$$

Whether you square a positive or a negative number, the answer is positive. It is impossible to square **any** number and get a negative answer.

So what is $\sqrt{-1}$? Since it asks the question “What number **squared** is –1?”, and since nothing squared ever gives the answer –1, we say that the question has no answer. More generally, we say that the domain of $\sqrt{x}$ is all numbers x such that $x \geq 0$. –1 is not in the domain.

However, it turns out that for a certain class of problems, it is useful to define a new kind of number that has the peculiar property that when you square them, you **do** get negative answers.

Definition of i

The definition of the imaginary number i is that it is the square root of –1:

$i = \sqrt{-1}$ or, equivalently, $i^2 + -1$

i is referred to as an “imaginary number” because it cannot represent real quantities such as “the number of rocks” or “the length of a stick.” However, surprisingly, imaginary numbers can be useful in solving many real world problems!

I often like to think of x as being like a science fiction story. Many science fiction stories are created by starting with **one false premise,** such as “time travel is possible” or “there are men on Mars,” and then following that premise logically to see where it would lead. With imaginary numbers, we start with the premise that “a number exists whose square is –1.” We then follow that premise logically, using all the established rules of math, to see where it leads.

" "The imaginary number is a fine and wonderful resource of the human spirit, almost an amphibian between being and not being."" "- Gottfried Wilhelm Leibniz"

Imaginary Concepts -- Playing with i
This module contains some example problems involving the manipulation i, the imaginary number.

Let’s begin with a few very simple exercises designed to show how we apply the normal rules of algebra to this new, abnormal number.

A few very simple examples of expressions involving i	
Simplify:	$i \bullet 5$
Answer:	$5\mathrm{i}$
Simplify:	$i + 5\mathrm{i}$
Answer:	$6\mathrm{i}$ (Add anything to 5 of itself, and you get 6 of it. Or, you can think of this as “pulling out” an i as follows: $i + 5\mathrm{i} = i(1+5) = 6\mathrm{i}$)
Simplify:	$2\mathrm{i} + 3$
Answer:	You can't simplify it.

Now let's try something a little more involved.

Example: Simplify the expression (3+2i)2	
$(3+2\mathrm{i})^2 = 3^2 + 2 \quad 3 \quad\quad 2\mathrm{i} \quad + (2\mathrm{i})^2$	**because**$(x+a)2 = x^2 + 2\mathrm{ax} + a^2$**as always**
$= 9 + 12\mathrm{i}–$	$(2\mathrm{i})^2 = (2\mathrm{i})(2\mathrm{i}) = (2)(2)(\mathrm{i})(\mathrm{i}) = 4\mathrm{i}^2 = –4$
$= 5 + 12\mathrm{i}$	**we can combine the** 9 **and** –4, **but not the** $12\mathrm{i}$.

It is vital to remember that i is not a variable, and this is not an algebraic generalization. You cannot plug $i = 3$ into that equation and expect anything valid to come out. The equation $(3+2\mathrm{i})^2 = 5 + 12\mathrm{i}$ has been shown to be true for only one number: that number is i, the square root of –1.

In the next example, we simplify a radical using **exactly the same technique that we used in the unit on radicals**, except that $a - 1$ is thrown into the picture.

Example: Simplify $\sqrt{-20}$	
$\sqrt{-20}$ = $\overline{(4)(5)(-1)}$	**as always, factor out the perfect squares**
= $\sqrt{4}\sqrt{5}\sqrt{-1}$	**then split it, because**$\sqrt{\mathrm{ab}}$= $\sqrt{a}\sqrt{b}$
$= 2\mathrm{i}\ \sqrt{5}$	$\sqrt{4}$=2, $\sqrt{-1}$= i, **and**$\sqrt{5}$**is just**$\sqrt{5}$
Check	
Is $2\mathrm{i}\ \sqrt{5}$ really the square root of –20? If it is, then when we **square** it, we should get –20.	
$2\mathrm{i}\sqrt{5}\ ^{2} = 2^2 i^2 5^2 = 4 * -1 * 5 = -20$ It works!	

The problem above has a very important consequence. We began by saying “You can’t take the square root of any negative number.” Then we defined i as the square root of –1. But we see that, using i, we can now take the square root of **any** negative number.

Imaginary Concepts -- Complex Numbers
This module introduces the concept of complex numbers in Algebra.

A “complex number” is the sum of two parts: a real number by itself, and a real number multiplied by . It can therefore be written as , where and are real numbers.

The first part, , is referred to as the **real part**. The second part, , is referred to as the **imaginary part**.

Examples of complex numbers (is the “real part”; is the “imaginary part”)	
	,
π	π, (no imaginary part: a “pure real number”)
	, (no real part: a “pure imaginary number”)

Some numbers are not obviously in the form . However, any number can be **put** in this form.

Example 1: Putting a fraction into form (in the numerator)
—— is a valid complex number. But it is not in the form , and we cannot immediately see what the real and imaginary parts are.

Example 1: Putting a fraction into form (in the numerator)
To see the parts, we rewrite it like this:
—— –– –
Why does that work? It’s just the ordinary rules of fractions, applied backward. (Try multiplying and then subtracting on the right to confirm this.) But now we have a form we can use:
—— – , –
So we see that fractions are very easy to break up, if the is in the numerator. An in the denominator is a bit trickier to deal with.

Example 2: Putting a fraction into form (in the denominator)	
– —	**Multiplying the top and bottom of a fraction by the same number never changes the value of the fraction: it just rewrites it in a different form.**
—	**Because is , or –1.**
	This is not a property of , but of –1. Similarly, — **.**

Example 2: Putting a fraction into form (in the denominator)	
–: ,	**since we rewrote it as , or**

Finally, what if the denominator is a more complicated complex number? The trick in this case is similar to the trick we used for rationalizing the denominator: we multiply by a quantity known as the **complex conjugate of the denominator**.

Definition of Complex Conjugate
The complex conjugate of the number is . In words, you leave the real part alone, and change the sign of the imaginary part.

Here is how we can use the “complex conjugate” to simplify a fraction.

Example: Using the Complex Conjugate to put a fraction into form	
——	**The fraction: a complex number not currently in the form**
——————	**Multiply the top and bottom by the complex conjugate of the denominator**
————	**Remember,**

Example: Using the Complex Conjugate to put a fraction into form	
———	**, which we are subtracting from 9**
———	**Success! The top has , but the bottom doesn’t. This is easy to deal with.**
— —	**Break the fraction up, just as we did in a previous example.**
– –	**So we’re there!** – and –

Any number of any kind can be written as . The above examples show how to rewrite fractions in this form. In the text, you go through a worksheet designed to rewrite as three different complex numbers. Once you understand this exercise, you can rewrite other radicals, such as , in form.

Imaginary Concepts -- Equality and Inequality in Complex Numbers

What does it mean for two complex numbers to be **equal?** As always, equality asserts that two things are exactly the same. is **not** equal to 7, or to , or to , or to . It is not equal to anything except .

Definition of Equality

Two complex numbers are **equal** to each other only if **their real parts are equal, and their imaginary parts are equal.**

So if we say that two complex numbers equal each other, we are actually making two separate, independent statements. We can use this, for instance, to solve for two separate variables.

Example: Complex Equality	
If , what are and ?	
Normally, it is impossible to solve one equation for two unknowns. But this is really two separate equations!	
Real part on the left = real part on the right:	
Imaginary part on the left = imaginary part on the right:	
We can now solve both of these equations trivially.	,

And what about **inequalities?** The answer may surprise you: **there are no inequalities with complex numbers,** at least not in the form we’re seeing.

The real numbers have the property that for any two real numbers and , exactly one of the following three statements must be true: , , or . This is one of those properties that seems almost too obvious to bother with. But it becomes more interesting when you realize that the complex numbers do **not** have that property. Consider two simple numbers, and 1. Which of the following is true?

- = 1
- > 1
- < 1

None of them is true. It is not generally possible to describe two complex numbers as being “greater than” or “less than” each other.

Visually, this corresponds to the fact that all the real numbers can be laid out on a number line: “greater than” means “to the right of” and so on. The complex numbers cannot be laid out on a number line. They are sometimes pictured on a 2-dimensional graph, where the real part is the coordinate and the imaginary part is the coordinate. But one point on a graph is neither greater than, nor less than, another point!

Imaginary Concepts -- Quadratic Equations and Complex Numbers

This module looks at quadratic equations with a negative discriminant in Algebra.

In the unit on quadratic equations and complex numbers, we saw that a quadratic equation can have **two answers, one answer, or no answers**.

We can now modify this third case. In cases where we described “no answers” there are actually two answers, but both are complex! This is easy to see if you remember that we found “no answers” when the **discriminant was negative**—that is, when the quadratic formula gave us a negative answer in the square root.

As an example, consider the equation:

$$2x^2 + 3x + 5 = 0$$

The quadratic equation gives us:

$$= \frac{-3 \pm \quad \overline{3^2 - 4(2)(5)}}{4} = \frac{-3 \pm \sqrt{-31}}{4}$$

This is the point where, in the “old days,” we would have given up and declared “no answer.” Now we can find two answers—both complex.

$$= \frac{-3}{4} \pm \frac{\sqrt{31}\sqrt{-1}}{4} = \frac{-3}{4} \pm \frac{\sqrt{31}}{4}$$

So we have two answers. Note that the two answers are **complex conjugates** of each other—this relationship comes directly from the quadratic formula.

Imaginary Concepts -- Extra for Experts
This is a module discussing several advanced concepts relating to complex numbers, including the differences between real and imaginary, and rational and irrational numbers.

Illegal Operations

So far, we have seen three different **illegal operations** in math.

1. You cannot take the square root of a negative number. (Hence, the domain of $\sqrt{x}$ is $x \geq$.)
2. You cannot divide by zero. (Hence, the domain of $\frac{1}{x}$ is $x \neq$.)
3. You cannot take the log of 0 or a negative number. (Hence, the domain of $\log(x)$ is $x >$.)

Imaginary numbers give us a way of violating the first restriction. Less obviously, they also give us a way of violating the third restriction: with imaginary numbers, you can take the log of a negative number.

So, how about that second restriction? Do you ever reach a point in math where the teacher admits “OK, we **really can** divide by 0 now”? Can we define a new imaginary number $j = \frac{1}{0}$?

The answer is emphatically no: **you really can’t divide by 0**. If you attempt to define an imaginary way around this problem, all of math breaks down. Consider the following simple example:

$5 \bullet 0 = 3 \bullet 0$	**That’s true**
$5 = 3$	**Divide both sides by 0**

You see? Dividing by 0 takes us from true conclusions to false ones.

The astonishing thing about the definition $i = \sqrt{-1}$ is that, although it is imaginary and nonsensical, it is **consistent:** it does not lead to any logical contradictions. You can find many ways to simplify $\frac{1}{i}$ and it will always reduce to –i in the end. Division by zero can never be consistent in this way, so it is always forbidden.

A great deal of Calculus is concerned with getting around this problem, by dividing by numbers that are **very close** to zero.

The World of Numbers

When you first learn about numbers, you learn the **counting numbers:**

1,2,3,4...	**counting numbers**

These numbers are perfect for answering questions such as “How many sticks do I have?” “How many days until Christmas?” “How many years old are you?”

For other questions, however, you run into limitations. In measuring temperature, for instance, we find that we need lower numbers than 1. Hence, we arrive at a broader list:

...–4,–3,–2,–1,0,1,2,3,4...	**integers**

The addition of 0 and the negative numbers gives us a new, broader set. The original idea of numbers is now seen as a special case of this more general idea; the original set is a subset of this one.

Still, if we are measuring lengths of sticks, we will find that often they fall **between** our numbers. Now we have to add fractions, or decimals, to create the set of **rational numbers**. I can no longer list the set, but I can give examples.

$\frac{1}{2}$, –3, $\frac{22}{7}$, 0, 2.718, 0.14141414...	**rational numbers**

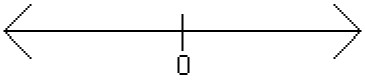
The word "rational" implies a **ratio**, or fraction: the ratio of **two integers**. Hence, we define our new, broader set (rational numbers) in terms of our older, more limited set (integers). Rational numbers can be expressed as either fractions, or as decimals (which either end after a certain number of digits, or repeat the same loop of digits forever).

This set seems to be all-inclusive, but it isn't: certain numbers cannot be expressed in this form.

$\sqrt{2}$, π	**irrational numbers**

The square root of any non-perfect square is "irrational" and so is π. They can be **approximated** as fractions, but not expressed exactly. As decimals, they go on forever but do not endlessly repeat the same loop.

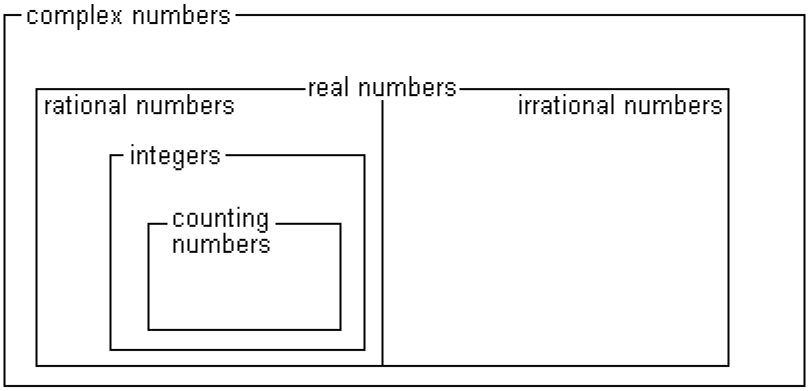
If you take the rationals and irrationals together, you get the **real numbers**. The real numbers are all the numbers represented on a number line.



All the numbers
on a number line
are the real
numbers

Now, with this unit, we have added the final piece of the puzzle, the **complex numbers**. A complex number is any number $a + \mathrm{bi}$ where a and b are real numbers. Hence, just as our definition of rational numbers was **based on** our definition of integers, so our definition of complex numbers is **based on** our definition of real numbers. And of course, if $b = 0$ then we have a real number: the old set is a subset of the new.

All of this can be represented in the following diagram.



The diagram captures the vital idea of subsets: all real numbers are complex numbers, but not all complex numbers are real.

Similarly, the diagram shows that if you take all the rational numbers, and all the irrational numbers, together they make up the set of real numbers.

Logarithm Concepts -- Introduction

This module introduces the concept of logarithms.

Suppose you are a biologist investigating a population that doubles every year. So if you start with 1 specimen, the population can be expressed as an exponential function: p t ^{t} where t is the number of years you have been watching, and p is the population.

Question: How long will it take for the population to exceed 1,000 specimens?

We can rephrase this question as: “2 to **what power** is 1,000?” This kind of question, where you know the base and are looking for the exponent, is called a **logarithm**.

(read, “the logarithm, base two, of a thousand”) means “2, raised to what power, is 1000?”

In other words, the logarithm always asks “**What exponent should we use**?” This unit will be an exploration of logarithms.

A few quick examples to start things off

Problem	Means	The answer is	because
	2 to what power is 8?	3	is 8
	2 to what power is	4	is 16

	16?		
	2 to what power is 10?	somewhere between 3 and 4	and
	8 to what power is 2?	–	– –
	10 to what power is 10,000?	4	
—	10 to what power is — ?	–2	— —
	5 to what power is 0?	There is no answer	will never be 0

As you can see, one of the most important parts of finding logarithms is being very familiar with how exponents work!

Logarithm Concepts -- The logarithm explained by analogy to roots

The logarithm may be the first really new concept you've encountered in Algebra II. So one of the easiest ways to understand it is by comparison with a familiar concept: roots.

Suppose someone asked you: "Exactly what does root mean?" You do understand roots, but they are difficult to define. After a few moments, you might come up with a definition very similar to the "question" definition of logarithms given above. $\sqrt[3]{8}$ means "what number cubed is 8?"

Now the person asks: "How do you find roots?" Well...you just play around with numbers until you find one that works. If someone asks for $\sqrt{25}$, you just have to know that $5^2 = 25$. If someone asks for $\sqrt{30}$, you know that has to be bigger than 5 and smaller than 6; if you need more accuracy, it's time for a calculator.

All that information about roots applies in a very analogous way to logarithms.

	Roots	**Logs**
The question	$\sqrt{\ }$ means "what number, raised to the a power, is x?" As an equation, $=$	log means " , raised to what power, is ?" As an equation, $=$
Example that comes out even	$\sqrt[3]{8} = 2$	$\log_2 8 = 3$

	Roots	**Logs**
Example that doesn't	$\sqrt[3]{10}$ is a bit more than 2	$\log_2 10$ is a bit more than 3
Out of domain example	$\sqrt{-4}$does not exist (^{2} will never give -4)	$\log_2(0)$ and $\log_2(-1)$ do not exist (2 will never give 0 or a negative answer)

Logarithm Concepts -- Rewriting logarithm equations as exponent equations

This module discusses how equations involving logarithms can be re-written using exponents.

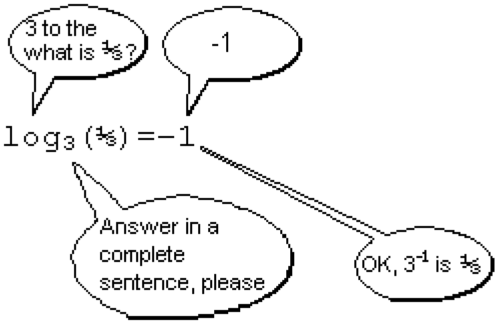
Both root equations and logarithm equations can be rewritten as exponent equations.

can be rewritten as . These two equations are the same statement about numbers, written in two different ways. asks the question “What number squared is 9?” So the equation asks this question, and then answers it: “3 squared is 9.”

We can rewrite logarithm equations in a similar way. Consider this equation:

Equation:

If you are asked to rewrite that logarithm equation as an exponent equation, think about it this way. The left side asks: “3 to what power is – ?” And the right side answers: “3 to the **power** is – .” – .


3 to the what is $\frac{1}{3}$?
-1
$\log_3(\frac{1}{3})=-1$
Answer in a complete sentence, please
OK, 3^{-1} is $\frac{1}{3}$

These two equations, – and – , are two different ways of expressing the same numerical relationship.

Logarithm Concepts -- The logarithm defined as an inverse function

This module discusses how logarithms can be defined as inverse functions.

$\sqrt{x}$ can be defined as the **inverse function** of x^2. Recall the definition of an inverse function— $f^{-1}(x)$ is defined as the inverse of $f^1(x)$ if it reverses the inputs and outputs. So we can demonstrate this inverse relationship as follows:

$\sqrt{x}$ is the inverse function of x^2
$3 \rightarrow x^2 \rightarrow 9$
$9 \rightarrow \sqrt{x} \rightarrow 3$

Similarly, $\log_2 x$ is the **inverse function** of the exponential function 2^x.

$\log_2 x$ is the inverse function of 2^x
$3 \rightarrow 2^x \rightarrow 8$
$8 \rightarrow \log_2 x \rightarrow 2$

(You may recall that during the discussion of inverse functions, 2^x was the only function you were given that you **could not find** the inverse of. Now you know!)

In fact, as we noted in the first chapter, $\sqrt{x}$is **not** a perfect inverse of x^2, since it does not work for negative numbers. $(-3)^2 = 9$, but $\sqrt{9}$ is not -3. Logarithms have no such limitation: $\log_2 x$ is a perfect inverse for 2^x.

The inverse of addition is subtraction. The inverse of multiplication is division. Why do exponents have two completely different kinds of inverses, roots and logarithms? **Because exponents do not commute**. 3^2 and 2^3 are not the same number. So the question “what number squared equals 10?” and the question “2 to what power equals 10?” are different questions, which we express as $\sqrt{10}$ and $\log_2 10$, respectively, and they have different answers. x^2 and 2^x are not the same function, and they therefore have different inverse functions $\sqrt{x}$ and $\log_2 10$.

Logarithm Concepts -- Properties of Logarithms

This module contains some of the properties of logarithms and how they can be used for manipulation.

Just as there are three fundamental laws of exponents, there are three fundamental laws of logarithms.

Equation:

Equation:

—

Equation:

As always, these **algebraic generalizations** hold for any , , and .

Example:

Properties of Logarithms

1. **Suppose you are given these two facts**:
    ○
    ○
2. **Then we can use the laws of logarithms to conclude that**:
    ○
    ○
    ○

Note:All three of these results can be found quickly, and without a calculator. Note that the second result could also be figured out directly, since .

These properties of logarithms were very important historically, because they enabled pre-calculator mathematicians to perform **multiplication** (which is very time-consuming and error prone) by doing **addition** (which is faster and easier). These rules are still useful in simplifying complicated expressions and solving equations.

Example:
Solving an equation with the properties of logarithms

	The problem
	Second property of logarithms
	Rewrite the log as an exponent. (2-to-what is? 2-to-the-5!)
	Multiply. We now have an easy equation to solve.

—	

Proving the Properties of Logarithms

If you understand what an exponent is, you can very quickly see why the three rules of exponents work. But why do logarithms have these three properties?

As you work through the text, you will demonstrate these rules intuitively, by viewing the logarithm as a **counter**. (asks “**how many** 2s do I need to multiply, in order to get 8?”) However, these rules can also be rigorously proven, using the laws of exponents as our starting place.

Proving the First Law of Logarithms,	
	I’m just inventing to represent this log
	Rewriting the above expression as an exponent. (asks “ to what power is ?” And the equation answers: “ to the is .”)
	Similarly, will represent the other log.

	Replacing and based on the previous equations
	This is the key step! It uses the first law of exponents. Thus you can see that the properties of logarithms come directly from the laws of exponents.
	asks the question: " to what power is ?" Looked at this way, the answer is obviously . Hence, you can see how the logarithm and exponential functions cancel each other out, as inverse functions must.
	Replacing and with what they were originally defined as. Hence, we have proven what we set out to prove.

To test your understanding, try proving the second law of logarithms: the proof is very similar to the first. For the third law, you need invent only one variable, . In each case, you will rely on a different one of the three rules of exponents, showing how each exponent law corresponds to one of the logarithms laws.

Logarithm Concepts -- Common Logarithms

This module covers some of the logarithms commonly encountered in algebra.

When you see a root without a number in it, it is assumed to be a **square** root. That is, $\sqrt{25}$is a shorthand way of writing $\sqrt[2]{25}$. This rule is employed because **square** roots are more common than other types.

When you see a logarithm without a number in it, it is assumed to be a **base 10** logarithm. That is, $\log(1000)$ is a shorthand way of writing $\log_{10}(1000)$. A base 10 logarithm is also known as a "common" log.

Why are common logs particularly useful? Well, what is $\log_{10}(1000)$? By now you know that this asks the question "10 to what power is 1000?" The answer is 3. Similarly, you can confirm that:

Equation:

$$\log(10) = 1$$

Equation:

$$\log(100) = 2$$

Equation:

$$\log(1{,}000\ 000) = 6$$

We can also follow this pattern backward:

Equation:

$$\log(1) = 0$$

Equation:

$$\log\left(\frac{1}{10}\right) = -1$$

Equation:

$$\log\left(\frac{1}{100}\right) = -2$$

and so on. In other words, the common log tells you the **order of magnitude** of a number: how many zeros it has. Of course, $\log_{10}(500)$ is difficult to determine exactly without a calculator, but we can say immediately that it must be somewhere between 2 and 3, since 500 is between 100 and 1000.

Logarithm Concepts -- Graphing Logarithmic Functions

This module discusses the graphing and plotting of logarithmic functions and some of their applications.

Suppose you want to graph the function $y = \log_2(x)$. You might start by making a table that looks something like this:

x	$y = \log_2(x)$
1	0
2	1
3	um....I’m not sure
4	2
5	can I use a calculator?

This doesn’t seem to be the right strategy. Many of those numbers are just too hard to work with.

So, you start looking for numbers that **are** easy to work with. And you remember that it’s important to look at numbers that are less than 1, as well as greater. And eventually, you end up with something more like this.

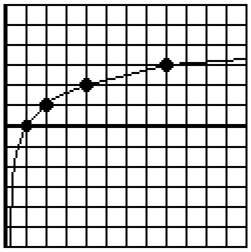
x	$y = \log_2(x)$

x	$y = \log_2(x)$
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

As long as you keep putting powers of 2 in the x column, the y column is very easy to figure.

In fact, the easiest way to generate this table is to recognize that it is the table of $y = 2^x$ values, only with the x and y coordinates switched! In other words, we have re-discovered what we already knew: that $y = 2^x$ and $y = \log_2(x)$ are inverse functions.

When you graph it, you end up with something like this:



$y = \log_2(x)$

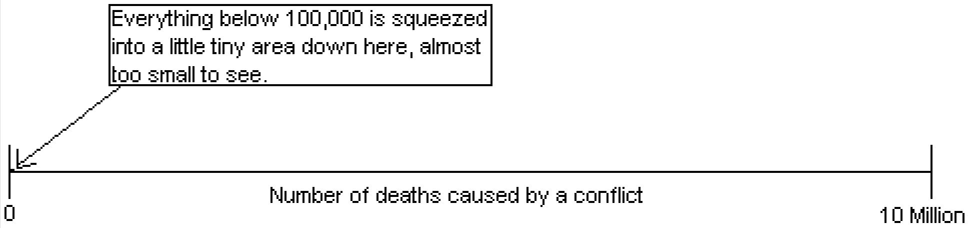
As always, you can learn a great deal about the log function by reading the graph.

- The domain is $x > 0$. (You can't take the log of 0 or a negative number—do you remember why?).
- The range, on the other hand, is all numbers. Of course, all this inverses the function 2^x, which has a **domain** of all numbers and a **range** of $y > 0$.
- As x gets closer and closer to 0, the function dives down to smaller and smaller negative numbers. So the y-axis serves as an "asymptote" for the graph, meaning a line that the graph **approaches closer and closer to** without ever touching.
- As x moves to the right, the graph grows—but more and more slowly. As x goes from 4 to 8, the graph goes up by 1. As x goes from 8 to 16, the graph goes up by another 1. It doesn't make it up another 1 until x reaches 32...and so on.

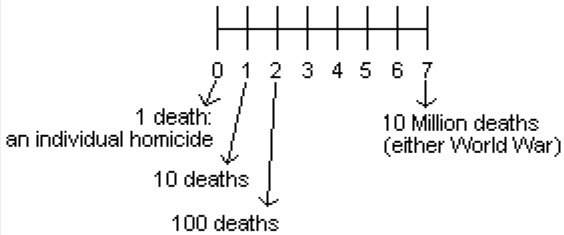
This pattern of **slower and slower growth** is one of the most important characteristics of the log. It can be used to "slow down" functions that have too wide a range to be practical to work with.

Example:
Using the log to model a real world problem

Lewis Fry Richardson (1881–1953) was a British meteorologist and mathematician. He was also an active Quaker and committed pacifist, and was one of the first men to apply statistics to the study of human conflict. Richardson catalogued 315 wars between 1820 and 1950, and categorized them by **how many deaths** they caused. At one end of the scale is a deadly quarrel, which might result in 1 or 2 deaths. At the other extreme are World War I and World War II, which are responsible for roughly 10 million deaths each.


Everything below 100,000 is squeezed into a little tiny area down here, almost too small to see.
Number of deaths caused by a conflict
0
10 Million

As you can see from the chart above, working with these numbers is extremely difficult: on a scale from 0 to 10 Million, there is no visible difference between (say) 1 and 100,000. Richardson solved this problem by taking the **common log of the number of deaths**. So a conflict with 1,000 deaths is given a magnitude of $\log(1000) = 3$. On this scale, which is now the standard for conflict measurement, the magnitudes of all wars can be easily represented.


0
1
2
3
4
5
6
7
1 death: an individual homicide
10 deaths
100 deaths
10 Million deaths (either World War)

Richardson's scale makes it practical to chart, discuss, and compare wars and battles from the smallest to the biggest. For instance, he discovered that each time you move up by one on the scale—that is, each time the number of deaths multiplies by 10—the number of conflicts drops in a third. (So there are roughly three times as many "magnitude 5" wars as "magnitude 6," and so on.)

The log is useful here because the logarithm function itself **grows so slowly** that it compresses the entire 1-to-10,000,000 range into a 0-to-7 scale. As you will see in the text, the same trick is used—for the same reason—in fields ranging from earthquakes to sound waves.

Simultaneous Equations Concepts -- Distance, Rate and Time

If you travel 30 miles per hour for 4 hours, how far do you go? A little common sense will tell you that the answer is 120 miles.

This relationship is captured in the following equation:

where...

- is distance traveled (sometimes the letter x is used instead, for position)
- is the rate, or speed (sometimes the letter v is used, for velocity)
- is the time

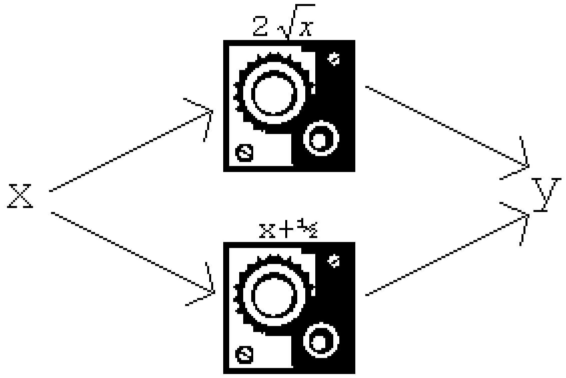
This is presented here because it forms the basis for many common simultaneous equations problems.

Simultaneous Equations Concepts -- Simultaneous Equations by Graphing

Consider the equation ‾. How many pairs are there that satisfy this equation? Answer: , , , and are all solutions; and there is an **infinite number of other solutions**. (And don't forget non-integer solutions, such as !)

Now, consider the equation –. How many pairs satisfy **this** equation? Once again, an infinite number. Most equations that relate two variables have an infinite number of solutions.

To consider these two equations "simultaneously" is to ask the question: what pairs make **both equations true**? To express the same question in terms of functions: what values can you hand the functions ‾ and – that will make these two functions produce the **same answer**?

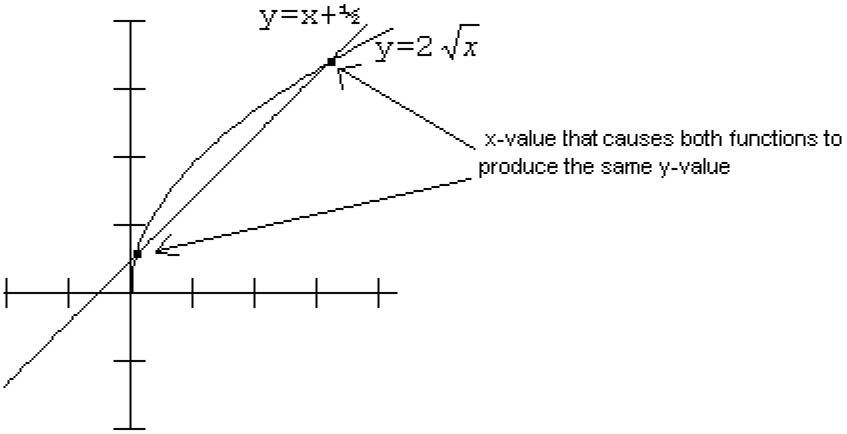


What number goes into **both functions** and makes them give the **same answer**? Is there even such a number? Is there more than one such number?

At first glance, it is not obvious how to approach such a question-- it is not even obvious how many answers there will be.

One way to answer such a question is by graphing. Remember, the graph of is the set of all points that satisfy that relationship; and the graph of is the set of all points that satisfy that relationship. So **the intersection(s) of these two graphs is the set of all points that satisfy both relationships**.

How can we graph these two? The second one is easy: it is a line, already in format. The -intercept is and the slope is 1. We can graph the first equation by plotting points; or, if you happen to know what the graph of looks like, you can stretch the graph vertically to get , since all the -values will double. Either way, you wind up with something like this:



We can see that there are two points of intersection. One occurs when is barely greater than 0 (say,), and the other occurs at approximately . There will be no more points of intersection after this, because the line will rise faster than the curve.

Exercise:

Problem:

Solution:

From graphing...

,

Graphing has three distinct advantages as a method for solving simultaneous equations.

1. It works on any type of equations.
2. It tells you **how many** solutions there are, as well as what the solutions are.
3. It can help give you an **intuitive feel** for why the solutions came out the way they did.

However, graphing also has two **dis**advantages.

1. It is time-consuming.
2. It often yields solutions that are **approximate**, not exact—because you find the solutions by simply “eyeballing” the graph to see where the two curves meet.

For instance, if you plug the number 3 into both of these functions, will you get the same answer?

– –

–

—

Pretty close! Similarly, , which is quite close to 0.6. But if we want more exact answers, we will need to draw a much more exact graph, which becomes **very** time-consuming. (Rounded to three decimal places, the actual answers are 0.086 and 2.914.)

For more exact answers, we use analytic methods. Two such methods will be discussed in this chapter: **substitution** and **elimination**. A third method will be discussed in the section on Matrices.

Simultaneous Equations Concepts -- Substitution

Here is the algorithm for substitution.

1. Solve one of the equations for one variable.
2. Plug this variable into the other equation.
3. Solve the second equation, which now has only one variable.
4. Finally, use the equation you found in step (1) to find the other variable.

Example:
Solving Simultaneous Equations by Substitution

$3x + 4y = 1$
$2x - \quad = 8$

1. The easiest variable to solve for here is the   in the second equation.
   - $- \quad = -2x + 8$
   - $\quad = 2x - 8$
2. Now, we plug that into the **other** equation:
   - $3x + 4(2x - 8) = 1$
3. We now have an equation with only   in it, so we can solve for   .
   - $3x + 8x - 32 = 1$
   - $11 \quad = 33$
   - $\quad = 3$
4. Finally, we take the equation from step (1), $\quad = 2x - 8$, and use it to find   .
   - $\quad = 2(3) - 8 = -2$

So $(3, -2)$ is the solution. You can confirm this by plugging this pair into both of the original equations.

Why does substitution work?

We found in the first step that $= 2x - 8$. This means that and $2x - 8$ are **equal** in the sense that we discussed in the first chapter on functions—they will always be the same number, in these equations—they are the **same**. This gives us permission to simply replace one with the other, which is what we do in the second (“substitution”) step.

Simultaneous Equations Concepts -- Elimination

Here is the algorithm for elimination.

1. Multiply one equation (or in some cases both) by some number, so that the two equations have the **same coefficient** for one of the variables.
2. Add or subtract the two equations to make that variable go away.
3. Solve the resulting equation, which now has only one variable.
4. Finally, plug back in to find the other variable.

Example:
Solving Simultaneous Equations by Elimination

- **1**The first question is: how do we get one of these variables to have the **same coefficient in both equations**? To get the coefficients to be the same, we would have to multiply the top equation by 2 and the bottom by 3. It is much easier with ; if we simply multiply the bottom equation by 4, then the two values will both be multiplied by 4.
    -
    -
- **2**Now we either **add** or **subtract** the two equations. In this case, we have on top, and on the bottom; so if we add them, they will cancel out. (If the bottom had we would have to subtract the two equations to get the " "s to cancel.)
    -
- **3-4** Once again, we are left with only one variable. We can solve this equation to find that and then plug back in to either of the original equations to find as before.

Why does elimination work?

As you know, you are always allowed to do the **same thing** to both sides of an equation. If an equation is true, it will still be true if you add 4 to both sides, multiply both sides by 6, or take the square root of both sides.

Now—consider, in the second step above, what we did to the equation . We **added** something to both sides of this equation. What did we add? On the left, we added ; on the right, we added 32. It seems that we have done something **different** to the two sides.

However, the second equation gives us a guarantee that these two quantities, and 32, are in fact **the same as each other**. So by adding to the left, and 32 to the right, we really have done exactly the same thing to both sides of the equation .

Simultaneous Equations Concepts -- Special Cases
This module discusses concepts related to simultaneous equations.

Consider the two equations:
Equation:

Equation:

Suppose we attempt to solve these two equations by elimination. So, we double the first equation and subtract, and the result is:
Equation:

Hey, what happened? 0 does not equal 13, no matter what is. Mathematically, we see that these two equations have **no** simultaneous solution. You asked the question “When will both of these equations be true?” And the math answered, “Hey, buddy, not until 0 equals 13.”

No solution.

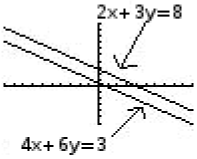
Now, consider these equations:
Equation:

Once again, we attempt elimination, but the result is different:
Equation:

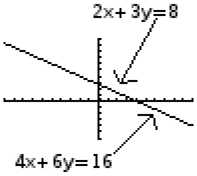
What happened **that** time? no matter what is. Instead of an equation that is always false, we have an equation that is always true. Does that mean these equations work for **any** and ? Clearly not: for instance, does not make either equation true. What this means is that the two equations are the same: any pair that solves one will also solve the other. There is an infinite number of solutions.

Infinite number of solutions.

All of this is much easier to understand graphically! Remember that one way to solve simultaneous equations is by graphing them and looking for the intersection. In the first case, we see that original equations represented two **parallel lines**. There is no point of intersection, so there is no simultaneous equation.



In the second case, we see that the original equations represented the **same line, in two different forms**. Any point on the line is a solution to both equations.



Note: If you solve an equation and get a mathematical impossibility such as , there is no solution. If you get a mathematical tautology such as , there is an infinite number of solutions.

Simultaneous Equations Concepts -- Word Problems

Many students approach math with the attitude that “I can do the equations, but I’m just not a ‘word problems’ person.” No offense, but that’s like saying “I’m pretty good at handling a tennis racket, as long as there’s no ball involved.” The only point of handling the tennis racket is to hit the ball. The only point of math equations is to solve problems. So if you find yourself in that category, try this sentence instead: “I’ve never been good at word problems. There must be something about them I don’t understand, so I’ll try to learn it.”

Actually, many of the key problems with word problems were discussed in the very beginning of the “Functions” unit, in the discussion of variable descriptions. So this might be a good time to quickly re-read that section. If you can correctly identify the variables, you’re half-way through the hard part of a word problem. The other half is **translating the sentences of the problem into equations** that use those variables.

Let’s work through an example, very carefully.

Example:
Simultaneous Equation Word Problem

A roll of dimes and a roll of quarters lie on the table in front of you. There are three more quarters than dimes. But the quarters are worth three times the amount that the dimes are worth. How many of each do you have?

1. Identify and label the variables.
    - There are actually two different, valid ways to approach this problem. You could make a variable that represents the number of dimes; or you could have a variable that represents the value of the dimes. Either way will lead you to the right answer. However, it is vital to know which one you’re doing! If you get confused half-way through the problem, you will end up with the wrong answer.

Let’s try it this way:

is the number of dimes

is the number of quarters

2. Translate the sentences in the problem into equations.

- “There are three more quarters than dimes”
- “The quarters are worth three times the amount that the dimes are worth”
- This second equation relies on the fact that if you have quarters, they are worth a total of cents.

3. Solve.

- We can do this by elimination or substitution. Since the first equation is already solved for , I will substitute that into the second equation and then solve.

So, did it work? The surest check is to go all the way back to the original problem—not the equations, but the words. We have concluded that there are 15 **dimes** and **18** quarters.

“There are three more quarters than dimes.”

“The quarters are worth three times the amount that the dimes are worth.”

Well, the quarters are worth . The dimes are worth .

Simultaneous Equations Concepts -- Using Letters as Numbers

This module shows how it can be helpful, on occasion, to use letters as numbers in order to quickly find solutions to variations on a simultaneous equation problem.

Toward the end of this chapter, there are some problems in substitution and elimination where **letters are used in place of numbers**. For instance, consider the following problem:

Equation:

$$2\text{y} - \text{ax} = 7$$

Equation:

$$4\text{y} + 3\text{ax} = 9$$

What do we do with those "a"s? Like any other variable, they simply represent an unknown number. As we solve for x, we will simply leave a as a variable.

This problem lends itself more naturally to elimination than to substitution, so I will double the top equation and then subtract the two equations and solve.

Equation:

$$\begin{gathered} 4\text{y} - 2\text{ax} = 14 \\ -(4\text{y} + 3\text{ax} = 9) \\ 0\text{y} - 5\text{ax} = 5 \end{gathered}$$

Equation:

$$x = \frac{5}{-5\text{a}} = \frac{-1}{a}$$

As always, we can solve for the second variable by plugging into either of our original equations.

Equation:

$$2\text{y} - a\left(\frac{-1}{a}\right) = 7$$

Equation:

$$2\text{y} + 1 = 7$$

Equation:

$$y = 3$$

There is no new math here, just elimination. The real trick is not to be spooked by the a, and do the math just like you did before.

And what does that **mean?** It means we have found a solution that works for those two equations, regardless of a. We can now solve the following three problems (and an infinite number of others) without going through the hard work.

If $a = 5$,	**If $a = 10$,**	**If $a = -3$,**
The original equations become:	The original equations become:	The original equations become:
$2\text{y} - 5\text{x} = 7$ $4\text{y} + 15x = 9$	$2\text{y} - 10x = 7$ $4\text{y} + 30x = 9$	$2\text{y} + 3\text{x} = 7$ $4\text{y} - 9\text{x} = 9$
And the solution is:	And the solution is:	And the solution is:

If $a = 5$**,**	**If** $a = 10$**,**	**If** $a = -3$**,**
$x = \frac{-1}{5}$ $y = 3$	$x = \frac{-1}{10}$ $y = 3$	$x = \frac{1}{3}$ $y = 3$

The whole point is that I did not have to **solve** those three problems—by elimination, substitution, or anything else. All I had to do was plug a into the general answer I had already found previously. If I had to solve a hundred such problems, I would have saved myself a great deal of time by going through the hard work once to find a general solution!

Mathematicians use this trick all the time. If they are faced with many similar problems, they will attempt to find a **general** problem that encompasses all the **specific** problems, by using variables to replace the numbers that change. You will do this in an even more general way in the text, when you solve the “general” simultaneous equations where **all** the numbers are variables. Then you will have a formula that you can plug any pair of simultaneous equations into to find the answer at once. This formula would also make it very easy, for instance, to program a computer to solve simultaneous equations (computers are terrible at figuring things out, but they’re great at formulas).

Matrix Concepts -- Explanations
This module introduces basic properties of matrices: concepts, multiplication by a constant, addition and subtraction, and setting two matrices equal to one another.

Conceptual Explanations: Matrices

A “matrix” is a grid, or table, of numbers. For instance, the following matrix represents the prices at the store “Nuthin’ But Bed Stuff.”

	King-sized	**Queen-sized**	**Twin**
Mattress	$649	$579	$500
Box spring	$350	$250	$200
Fitted sheet	$15	$12	$10
Top sheet	$15	$12	$10
Blanket	$20	$20	$15

(The matrix is the numbers, not the words that label them.)

Of course, these prices could be displayed in a simple list: “King-sized mattress,” “Queen-sized mattress,” and so on. However, this two-dimensional display makes it much easier to compare the prices of mattresses to box springs, or the prices of king-sized items to queen-sized items, for instance.

Each horizontal list of numbers is referred to as a **row;** each vertical list is a **column.** Hence, the list of all mattresses is a row; the list of all king-sized prices is a column. (It’s easy to remember which is which if you think of Greek columns, which are big posts that hold up buildings and are very tall and...well, you know...vertical.) This particular matrix has 5 rows and 3 columns. It is therefore referred to as a 5×3 (read, “5 by 3”) matrix.

If a matrix has the same number of columns as rows, it is referred to as a **square matrix**.

Adding and Subtracting Matrices

Adding matrices is very simple. You just add each number in the first matrix, to the corresponding number in the second matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 60 & 50 & 40 \\ 30 & 20 & 10 \end{bmatrix} = \begin{bmatrix} 61 & 52 & 43 \\ 34 & 25 & 16 \end{bmatrix}$$

For instance, for the upper-right-hand corner, the calculation was $3 + 40 = 43$. Note that both matrices being added are 2×3, and the resulting matrix is also 2×3. **You cannot add two matrices that have different dimensions.**

As you might guess, subtracting works much the same way, except that you subtract instead of adding.

$$\begin{bmatrix} 60 & 50 & 40 \\ 30 & 20 & 10 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 59 & 48 & 37 \\ 26 & 15 & 4 \end{bmatrix}$$

Once again, note that the resulting matrix has the same dimensions as the originals, and that **you cannot subtract two matrices that have different dimensions.**

Multiplying a Matrix by a Constant

What does it mean to multiply a number by 3? It means you add the number to itself 3 times.

Multiplying a **matrix** by 3 means the same thing...you add the matrix to itself 3 times.

$$3\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}=\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}+\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}+\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}=\begin{bmatrix}3 & 6 & 9\\12 & 15 & 18\end{bmatrix}$$

Note what has happened: each element in the original matrix has been multiplied by 3. Hence, we arrive at the method for multiplying a matrix by a constant: you multiply each element by that constant. The resulting matrix has the same dimensions as the original.

$$\tfrac{1}{2}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}=\begin{matrix}\frac{1}{2} & 1 & \frac{3}{2}\\2 & \frac{5}{2} & 3\end{matrix}$$

Matrix Equality

For two matrices to be “equal” they must be exactly the same. That is, they must have the same dimensions, and each element in the first matrix must be equal to the corresponding element in the second matrix.

For instance, consider the following matrix equation.

$$\begin{bmatrix}1 & + \\12 & 10\end{bmatrix}=\begin{bmatrix}1 & 18\\ - & 10\end{bmatrix}$$

Both matrices have the same dimensions. And the upper-left and lower-right elements are definitely the same.

But for the matrix to be equal, we also need the other two elements to be the same. So

$$+ \quad = 18$$

$$- \quad = 12$$

Solving these two equations (for instance, by elimination) we find that $= 15, \quad = 3$.

You may notice an analogy here to complex numbers. When we assert that two complex numbers equal each other, we are actually making two statements: the **real** parts are equal, and the **imaginary** parts are equal. In such a case, we can use one equation to solve for two unknowns. A very similar situation exists with matrices, except that one equation actually represents **many more** statements. For 2×2 matrices, setting them equal makes four separate statements; for 2×3 matrices, six separate statements; and so on.

OK, take a deep breath. Even if you've never seen a matrix before, the concept is not too difficult, and everything we've seen so far should be pretty simple, if not downright obvious.

Let that breath out now. This is where it starts to get weird.

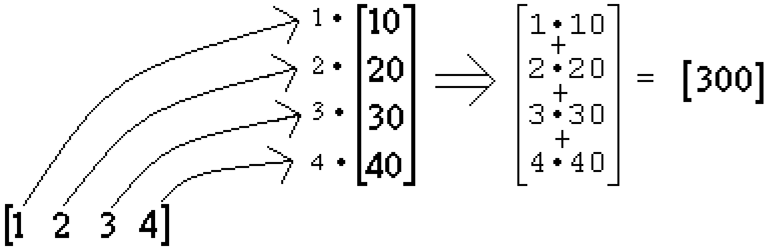
Matrix Concepts -- Multiplying Matrices

This module covers multiplication of matrices.

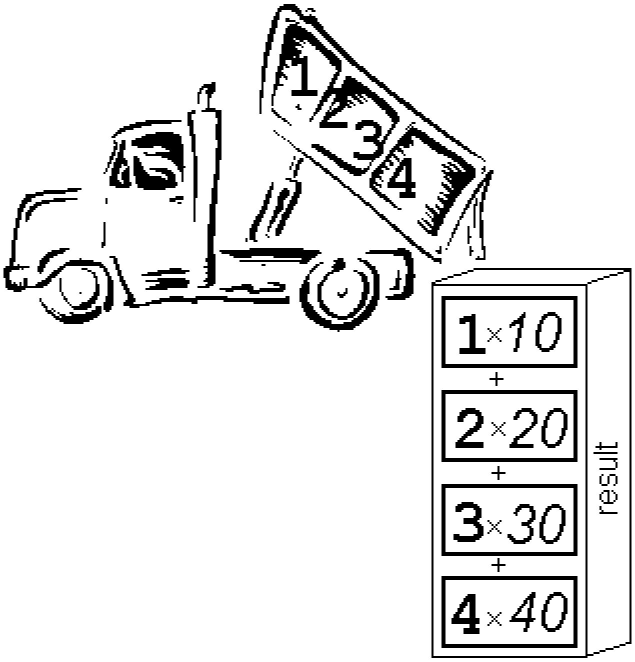
Multiplying a Row Matrix by a Column Matrix

A “row matrix” means a matrix with only one row. A “column matrix” means a matrix with only one column. When a row matrix has the same number of elements as a column matrix, they can be multiplied. So the following is a perfectly **legal** matrix multiplication problem:

These two matrices could not be added, of course, since their dimensions are different, but they **can** be multiplied. Here’s how you do it. You multiply the first (left-most) item in the row, by the first (top) item in the column. Then you do the same for the second items, and the third items, and so on. Finally, you **add** all these products to produce the final number.



A couple of my students (Nakisa Asefnia and Laura Parks) came up with an ingenious trick for visualizing this process. Think of the row as a dump truck, backing up to the column dumpster. When the row dumps its load, the numbers line up with the corresponding numbers in the column, like so:



So, without the trucks and dumpsters, we express the result—a row matrix, times a column matrix—like this:

There are several subtleties to note about this operation.

- The picture is a bit deceptive, because it might appear that you are multiplying two columns. In fact, **you cannot multiply a column matrix by a column matrix**. We are multiplying a row matrix by a column matrix. The picture of the **row** matrix "dumping down" only demonstrates which numbers to multiply.
- The answer to this problem is not a number: it is a 1-by-1 matrix.
- The multiplication can only be performed if the **number of elements** in each matrix is the same. (In this example, each matrix has 4 elements.)
- Order matters! We are multiplying a **row matrix times a column matrix**, not the other way around.

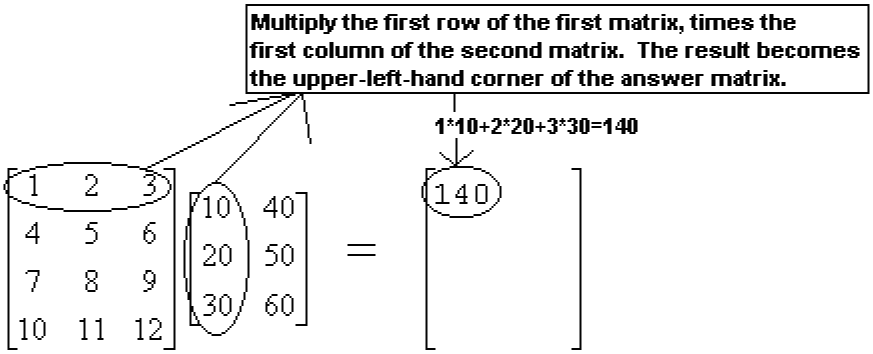
It’s important to practice a few of these, and get the hang of it, before you move on.

Multiplying Matrices in General

The general algorithm for multiplying matrices is built on the row-times-column operation discussed above. Consider the following example:

The key to such a problem is to think of the first matrix as a list of **rows** (in this case, 4 rows), and the second matrix as a list of **columns** (in this case, 2 columns). You are going to multiply each **row** in the first matrix, by each **column** in the second matrix. In each case, you will use the “dump truck” method illustrated above.

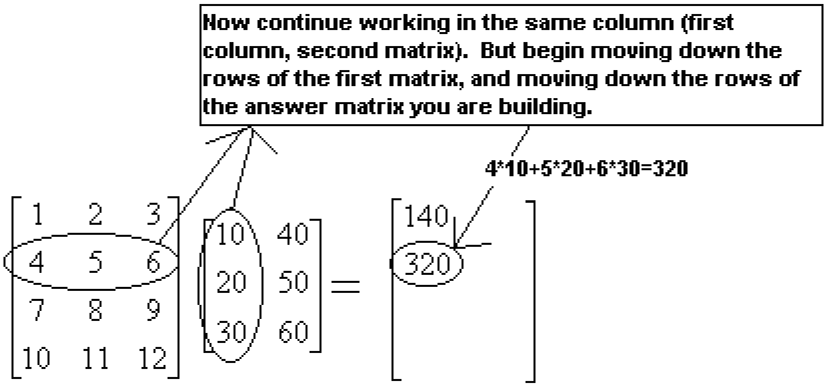
Start at the beginning: first row, times first column.



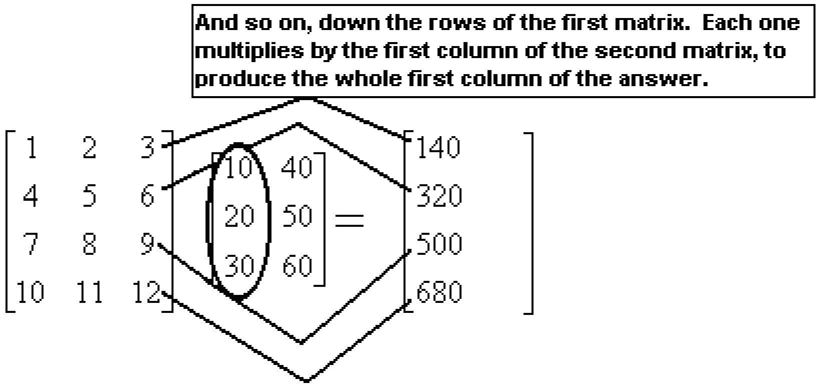
Multiply the first row of the first matrix, times the first column of the second matrix. The result becomes the upper-left-hand corner of the answer matrix.

1*10+2*20+3*30=140

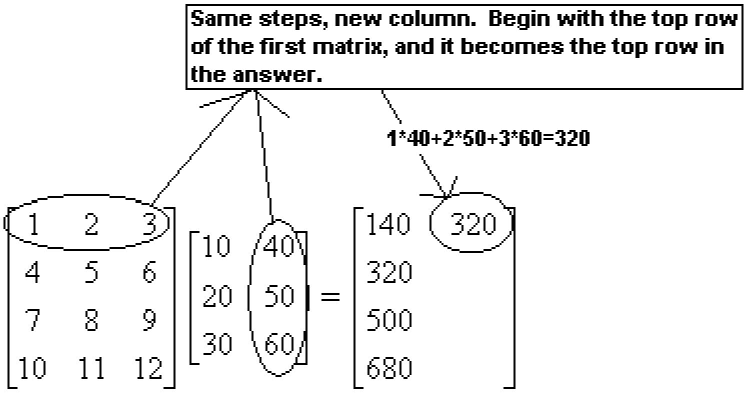
Now, move **down** to the next row. As you do so, move **down** in the answer matrix as well.


Now continue working in the same column (first column, second matrix). But begin moving down the rows of the first matrix, and moving down the rows of the answer matrix you are building.
4*10+5*20+6*30=320
1 2 3
4 5 6
7 8 9
10 11 12
10 40
20 50
30 60
=
140
320

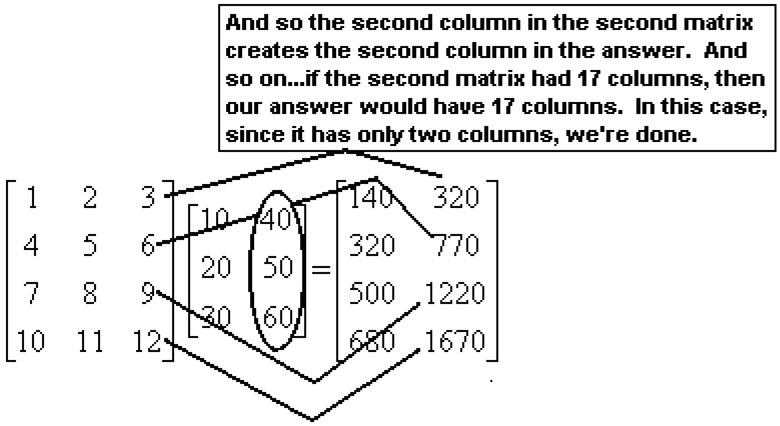
Now, move down the rows in the first matrix, multiplying each one by that same column on the right. List the numbers below each other.


And so on, down the rows of the first matrix. Each one multiplies by the first column of the second matrix, to produce the whole first column of the answer.
1 2 3
4 5 6
7 8 9
10 11 12
10 40
20 50
30 60
=
140
320
500
680

The **first column** of the second matrix has become the **first column** of the answer. We now move on to the **second column** and repeat the entire process, starting with the first row.


Same steps, new column. Begin with the top row of the first matrix, and it becomes the top row in the answer.
1*40+2*50+3*60=320
1 2 3
4 5 6
7 8 9
10 11 12
10 40
20 50
30 60
=
140 320
320
500
680

And so on, working our way once again through all the rows in the first matrix.



And so the second column in the second matrix creates the second column in the answer. And so on...if the second matrix had 17 columns, then our answer would have 17 columns. In this case, since it has only two columns, we're done.

We're done. We can summarize the results of this entire operation as follows:

It's a strange and ugly process—but everything we're going to do in the rest of this unit builds on this, so it's vital to be comfortable with this process. **The only way to become comfortable with this process is to do it. A lot.** Multiply a lot of matrices until you are confident in the steps.

Note that we could add more rows to the first matrix, and that would add more rows to the answer. We could add more **columns** to the second matrix, and that would add more columns to the answer. However—if we added a column to the first matrix, or added a row to the second matrix, we would have an illegal multiplication. As an example, consider what happens if we try to do this multiplication in reverse:

Illegal multiplication

If we attempt to multiply these two matrices, we start (as always) with the first row of the first matrix, times the first column of the second matrix:

. But this is an illegal multiplication; the items don’t line up,

since there are two elements in the row and four in the column. So you cannot multiply these two matrices.

This example illustrates two vital properties of matrix multiplication.

- The number of columns in the first matrix, and the number of rows in the second matrix, must be equal. Otherwise, you cannot perform the multiplication.
- Matrix multiplication is not **commutative**—which is a fancy way of saying, order matters. If you reverse the order of a matrix multiplication, you may get a different answer, or you may (as in this case) get no answer at all.

Matrix Concepts -- The Identity Matrix

This module introduces the identity matrix and its properties.

When multiplying numbers, the number 1 has a special property: when you multiply 1 by any number, you get that same number back. We can express this property as an algebraic generalization:

Equation:

The matrix that has this property is referred to as the **identity matrix**.

Definition of Identity Matrix

The **identity matrix**, designated as , is defined by the property:

Note that the definition of [I] stipulates that the multiplication must **commute**—that is, it must yield the same answer no matter which order you multiply in. This is important because, for most matrices, multiplication does not commute.

What matrix has this property? Your first guess might be a matrix full of 1s, but that doesn't work:

=	**so is not an identity matrix**

The matrix that **does** work is a diagonal stretch of 1s, with all other elements being 0.

=	**so is the identity for 2x2 matrices**
=	**is the identity for 3x3 matrices**

You should confirm those multiplications for yourself, and also confirm that they work in reverse order (as the definition requires).

Hence, we are led from the definition to:

The Identity Matrix
For any square matrix, its identity matrix is a diagonal stretch of 1s going from the upper-left-hand corner to the lower-right, with all other elements being 0. **Non-square** matrices do not have an identity. That is, for a non-square matrix , there is no matrix such that .

Why no identity for a non-square matrix? Because of the requirement of commutativity. For a non-square matrix you might be able to find a matrix such that ; however, if you reverse the order, you will be left with an illegal multiplication.

Matrix Concepts -- The Inverse Matrix

This module introduces the inverse matrix and its properties.

We have seen that the number 1 plays a special role in multiplication, because .

The **inverse of a number** is defined as the number that multiplies by that number to give 1: is the inverse of if . Hence, the inverse of is —; the inverse of — —. Every number except 0 has an inverse.

By analogy, the **inverse of a matrix** multiplies by that matrix to give the identity matrix.

Definition of Inverse Matrix

The **inverse of matrix** , designated as , is defined by the property:

The superscript –1 is being used here in a similar way to its use in functions. Recall that does not designate an exponent of any kind, but instead, an inverse function. In the same way, does not denote an exponent, but an inverse **matrix**.

Note that, just as in the definition of the identity matrix, this definition requires commutativity—the multiplication must work the same in either order.

Note also that only square matrices can have an inverse. Why? The definition of an inverse matrix is **based on** the identity matrix , and we already said that only square matrices even **have** an identity!

How do you **find** an inverse matrix? The method comes directly from the definition, with a little algebra.

Example: Finding an Inverse Matrix	
Find the inverse of	**The problem**

Example: Finding an Inverse Matrix	
=	**This is the key step. It establishes as the inverse that we are looking for, by asserting that it fills the definition of an inverse matrix: when you multiply this mystery matrix by our original matrix, you get [I]. When we solve for the four variables a, b, c, and d, we will have found our inverse matrix.**

Example: Finding an Inverse Matrix	
=	**Do the multiplication. (You should check this step for yourself, it’s great practice. For instance, you start by multiplying first row x first column, and you get 3a+4c.)**
	Remember what it means for two matrices to be equal: every element in the left must equal its corresponding element on the right. So, for these two matrices to equal each other, all four of these equations must hold.

Example: Finding an Inverse Matrix	
– –	**Solve the first two equations for a and c by using either elimination or substitution. Solve the second two equations for and by using either elimination or substitution. (The steps are not shown here.)**
So the inverse is: – –	**Having found the four variables, we have found the inverse.**

Did it work? Let’s find out.

Testing our Inverse Matrix	
– – $\overset{?}{=}$	**The definition of an inverse matrix: if we have indeed found an inverse, then when we multiply it by the original matrix, we should get [I].**

Testing our Inverse Matrix	
— — — — $\overset{?}{=}$	**Do the multiplication.**
$=$ — — ☑	**It works!**

Note that, to fully test it, we would have to try the multiplication in **both orders**. Why? Because, in general, changing the order of a matrix multiplication changes the answer; but the definition of an inverse matrix specifies that it must work both ways! Only one order was shown above, so technically, we have only half-tested this inverse.

This process does not have to be memorized: it should make logical sense. Everything we have learned about matrices should make logical sense, except for the very arbitrary-looking definition of matrix multiplication.

Matrix Concepts -- Determinants
This module covers matrix determinants and their uses.

The Determinant of a 2x2 Matrix

In the exercise “Inverse of the Generic 2x2 Matrix,” you found that the inverse of the matrix is —— . This formula can be used to very quickly find the inverse of any 2x2 matrix.

Note that if , the formula does not work, since it puts a 0 in the denominator. This tells us that, for any 2x2 matrix, if the matrix **has no inverse**.

The quantity ad–bc is therefore seen to have a special importance for 2x2 matrices, and it is accorded a special name: the “determinant.” Determinants are represented mathematically with absolute value signs: the determinant of matrix [A] is |A|.

Definition of the Determinant of a 2x2 Matrix

If matrix , the determinant is the number .

For instance, for the matrix , the determinant is (3)(6)–(4)(5) = –2.

Note that the determinant is a **number**, not a matrix. It is a special number that is **associated with** a matrix.

We said earlier that “if the matrix **has no inverse**.” We can now restate this result.

Any square matrix whose determinant is **not** 0, has an inverse matrix. Any square matrix with determinant 0 has no inverse.

This very important result is analogous to the result stated earlier for numbers: every number except 0 has an inverse.

The Determinant of a 3x3 Matrix (or larger)

Any **square matrix** has a determinant—an important number associated with that matrix. Non-square matrices do not have a determinant.

How do you find the determinant of a 3x3 matrix? The method presented here is referred to as “expansion by minors.” There are other methods, but they turn out to be mathematically equivalent to this one: that is, they end up doing the same arithmetic and arriving at the same answer.

Example: Finding the Determinant of a 3x3 Matrix	
Find the determinant of	**The problem.**
SORRY, THIS MEDIA TYPE IS NOT SUPPORTED.	**We’re going to walk through the top row, one element at a time, starting with the first element (the 2). In each case, begin by crossing out the row and column that contain that number.**
=(8)(1)–(3)(1)=5	**Once you cross out one row and column, you are left with a 2x2 matrix (a “minor”). Take the determinant of that matrix.**
2(5)=10	**Now, that “minor” is what we got by crossing out a 2 in the top row. Multiply that number in the top row (2) by the determinant of the minor (5).**

Example: Finding the Determinant of a 3x3 Matrix	
SORRY, THIS MEDIA TYPE IS NOT SUPPORTED. (10)(1)–(3)(1)= 74(7)=28	**Same operation for the second element in the row (the 4 in this case)...**
SORRY, THIS MEDIA TYPE IS NOT SUPPORTED. (10)(1)–(8)(1) = 25(2)=10	**...and the third (the 5 in this case).**
+10 – 28 + 10 = –8	**Take these numbers, and alternately add and subtract them; add the first, subtract the second, add the third. The result of all that is the determinant.**

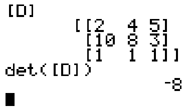
This entire process can be written more concisely as:

This method of “expansion of minors” can be extended upward to any higher-order square matrix. For instance, for a 4x4 matrix, each “minor” that is left when you cross out a row and column is a 3x3 matrix. To find the determinant of the 4x4, you have to find the determinants of all four 3x3 minors!

Fortunately, your calculator can also find determinants. Enter the matrix given above as matrix [D]. Then type:

`MATRX` ▶ `1MATRX 4 ) ENTER`

The screen should now look like this:



```
[D]
      [[2  4 5]
       [10 8 3]
       [1  1 1]]
det([D])
             -8
█
```

If you watched the calculator during that sequence, you saw that the right-arrow key took you to the `MATH` submenu within the `MATRIX` menus. The first item in that submenu is `DET (` which means “determinant of.”

What does the determinant mean? It turns out that this particular odd set of operations has a surprising number of applications. We have already seen one—in the case of a 2x2 matrix, the determinant is part of the inverse. And for any square matrix, the determinant tells you whether the matrix **has** an inverse at all.

Another application is for finding the area of triangles. To find the area of a triangle whose vertices are (a,b), (c,d), and (e,f), you can use the formula: Area =

½ . Hence, if you draw a triangle with vertices (2,10), (4,8), and (5,3),

the above calculation shows that the area of this triangle will be 4.

Definitions, Simple Applications, and Graphs of Trigonometric Functions

Introduction

In geometry we learn about how the sides of polygons relate to the angles in the polygons, but we have not learned how to calculate an angle if we only know the lengths of the sides. Trigonometry (pronounced: trig-oh-nom-eh-tree) deals with the relationship between the angles and the sides of a right-angled triangle. We will learn about trigonometric functions, which form the basis of trigonometry.

Investigation : History of Trigonometry

Work in pairs or groups and investigate the history of the foundation of trigonometry. Describe the various stages of development and how the following cultures used trigonometry to improve their lives.

The works of the following people or cultures can be investigated:

1. Cultures
   a. Ancient Egyptians
   b. Mesopotamians
   c. Ancient Indians of the Indus Valley
2. People
   a. Lagadha (circa 1350-1200 BC)
   b. Hipparchus (circa 150 BC)
   c. Ptolemy (circa 100)
   d. Aryabhata (circa 499)
   e. Omar Khayyam (1048-1131)
   f. Bhaskara (circa 1150)
   g. Nasir al-Din (13th century)
   h. al-Kashi and Ulugh Beg (14th century)
   i. Bartholemaeus Pitiscus (1595)

Note:
Interesting Fact
You should be familiar with the idea of measuring angles from geometry but have you ever stopped to think why there are 360 degrees in a circle? The reason is purely historical. There are 360 degrees in a circle because the ancient Babylonians had a number system with base 60. A base is the number at which you add another digit when you count. The number system that we use everyday is called the decimal system (the base is 10), but computers use the binary system (the base is 2). $360 = 6 \times 60$ so for them it made sense to have 360 degrees in a circle.

Where Trigonometry is Used

There are many applications of trigonometry. Of particular value is the technique of triangulation, which is used in astronomy to measure the distance to nearby stars, in geography to measure distances between landmarks, and in satellite navigation systems. GPSs (global positioning systems) would not be possible without trigonometry. Other fields which make use of trigonometry include astronomy (and hence navigation, on the oceans, in aircraft, and in space), music theory, acoustics, optics, analysis of financial markets, electronics, probability theory, statistics, biology, medical imaging (CAT scans and ultrasound), pharmacy, chemistry, number theory (and hence cryptology), seismology, meteorology, oceanography, many physical sciences, land surveying and geodesy, architecture, phonetics, economics, electrical engineering, mechanical engineering, civil engineering, computer graphics, cartography, crystallography and game development.

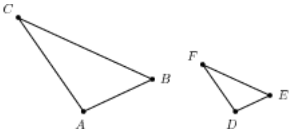
Discussion : Uses of Trigonometry

Select one of the uses of trigonometry from the list given and write a 1-page report describing *how* trigonometry is used in your chosen field.

Similarity of Triangles

If $\triangle ABC$ is similar to $\triangle DEF$, then this is written as:
Equation:

$$\triangle ABC \sim \triangle DEF$$



Then, it is possible to deduce ratios between corresponding sides of the two triangles, such as the following:
Equation:

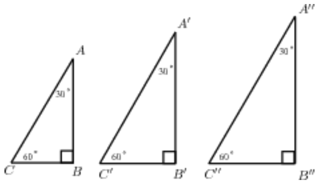
$$\begin{aligned} \frac{AB}{BC} &= \frac{DE}{EF} \\ \frac{AB}{AC} &= \frac{DE}{DF} \\ \frac{AC}{BC} &= \frac{DF}{EF} \\ \frac{AB}{DE} &= \frac{BC}{EF} = \frac{AC}{DF} \end{aligned}$$

The most important fact about similar triangles ABC and DEF is that the angle at vertex A is equal to the angle at vertex D, the angle at B is equal to the angle at E, and the angle at C is equal to the angle at F.
Equation:

$$\begin{aligned} \angle A &= \angle D \\ \angle B &= \angle E \\ \angle C &= \angle F \end{aligned}$$

Investigation : Ratios of Similar Triangles

In your exercise book, draw three similar triangles of different sizes, but each with $\widehat{A} = 30^\circ$; $\widehat{B} = 90^\circ$ and $\widehat{C} = 60^\circ$. Measure angles and lengths very accurately in order to fill in the table below (round answers to one decimal place).



Dividing lengths of sides (Ratios)		
$\frac{AB}{BC} =$	$\frac{AB}{AC} =$	$\frac{CB}{AC} =$
$\frac{A'B'}{B'C'} =$	$\frac{A'B'}{A'C'} =$	$\frac{C'B'}{A'C'} =$
$\frac{A''B''}{B''C''} =$	$\frac{A''B''}{A''C''} =$	$\frac{C''B''}{A''C''} =$

What observations can you make about the ratios of the sides?

These equal ratios are used to define the trigonometric functions.

Note: In algebra, we often use the letter x for our unknown variable (although we can use any other letter too, such as a, b, k, etc). In trigonometry, we often use the Greek symbol θ for an unknown angle (we also use α , β , γ etc).

Definition of the Trigonometric Functions

We are familiar with a function of the form $f(x)$ where f is the function and x is the argument. Examples are:
Equation:

$$\begin{aligned} f(x) &= 2^x && \text{(exponential function)} \\ g(x) &= x+2 && \text{(linear function)} \\ h(x) &= 2x^2 && \text{(parabolic function)} \end{aligned}$$

The basis of trigonometry are the *trigonometric functions*. There are three basic trigonometric functions:

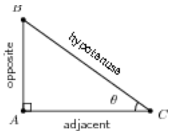
1. sine
2. cosine
3. tangent

These are abbreviated to:

1. sin
2. cos
3. tan

These functions are defined from a **right-angled triangle**, a triangle where one internal angle is 90 °.

Consider a right-angled triangle.



In the right-angled triangle, we refer to the lengths of the three sides according to how they are placed in relation to the angle θ. The side opposite to the right angle is labelled the *hypotenus*, the side opposite θ is labelled *opposite*, the side next to θ is labelled *adjacent*. Note that the choice of non-90 degree internal angle is arbitrary. You can choose either internal angle and then define the adjacent and opposite sides accordingly. However, the hypotenuse remains the same regardless of which internal angle you are referring to.

We define the trigonometric functions, also known as trigonometric identities, as:
Equation:

$$\begin{aligned} \sin \theta &= \frac{opposite}{hypotenuse} \\ \cos \theta &= \frac{adjacent}{hypotenuse} \\ \tan \theta &= \frac{opposite}{adjacent} \end{aligned}$$

These functions relate the lengths of the sides of a right-angled triangle to its interior angles.

One way of remembering the definitions is to use the following mnemonic that is perhaps easier to remember:

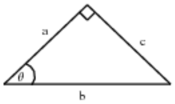
Silly **O**ld **H**ens	$\mathbf{S}\text{in} = \frac{\mathbf{O}\,\text{pposite}}{\mathbf{H}\,\text{ypotenuse}}$
Cackle **A**nd **H**owl	$\mathbf{C}\text{os} = \frac{\mathbf{A}\,\text{djacent}}{\mathbf{H}\,\text{ypotenuse}}$
Till **O**ld **A**ge	$\mathbf{T}\text{an} = \frac{\mathbf{O}\,\text{pposite}}{\mathbf{A}\,\text{djacent}}$

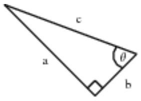
You may also hear people saying Soh Cah Toa. This is just another way to remember the trig functions.

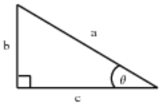
Note: The definitions of opposite, adjacent and hypotenuse are only applicable when you are working with right-angled triangles! Always check to make sure your triangle has a right-angle before you use them, otherwise you will get the wrong answer. We will find ways of using our knowledge of right-angled triangles to deal with the trigonometry of non right-angled triangles in Grade 11.

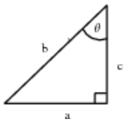
Investigation : Definitions of Trigonometric Functions

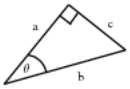
1. In each of the following triangles, state whether a, b and c are the hypotenuse, opposite or adjacent sides of the triangle with respect to the marked angle.

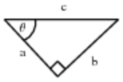




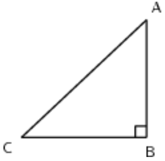








2. Complete each of the following, the first has been done for you



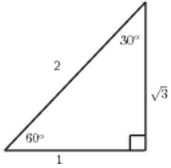
Equation:

$$\begin{aligned} a) \quad \sin \widehat{A} &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{CB}{AC} \\ b) \quad \cos \widehat{A} &= \\ c) \quad \tan \widehat{A} &= \end{aligned}$$

Equation:

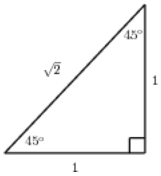
$$\begin{aligned} d) &\quad \sin \widehat{C} = \\ e) &\quad \cos \widehat{C} = \\ f) &\quad \tan \widehat{C} = \end{aligned}$$

3. Complete each of the following without a calculator:



Equation:

$$\begin{aligned} \sin 60 &= \\ \cos 30 &= \\ \tan 60 &= \end{aligned}$$



Equation:

$$\begin{aligned} \sin 45 &= \\ \cos 45 &= \\ \tan 45 &= \end{aligned}$$

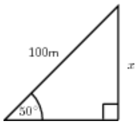
For most angles θ, it is very difficult to calculate the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$. One usually needs to use a calculator to do so. However, we saw in the above Activity that we could work these values out for some special angles. Some of these angles are listed in the table below, along with the values of the trigonometric functions at these angles. Remember that the lengths of the sides of a right angled triangle must obey Pythagoras' theorum. The square of the hypothenuse (side opposite the 90 degree angle) equals the sum of the squares of the two other sides.

	0°	30°	45°	60°	90°	180°
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$-$	0

These values are useful when asked to solve a problem involving trig functions *without* using a calculator.

Exercise:
Finding Lengths

Problem: Find the length of x in the following triangle.



Solution:

Identify the trig identity that you need In this case you have an angle (50°), the opposite side and the hypotenuse. So you should use sin **Equation:**

$$\sin 50^\circ = \frac{x}{100}$$

Rearrange the question to solve for **Equation:**

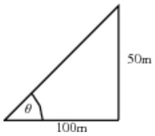
$$\Rightarrow x = 100\times \sin 50^\circ$$

Use your calculator to find the answer Use the sin button on your calculator **Equation:**

$$\Rightarrow x = 76.6\text{m}$$

Exercise:
Finding Angles

Problem: Find the value of θ in the following triangle.



Solution:

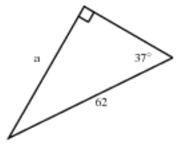
Identify the trig identity that you need	In this case you have the opposite side and the hypotenuse to the angle θ.	So you should use $\tan$		**Equation:** $\tan\theta = \frac{50}{100}$
Calculate the fraction as a decimal number				**Equation:** $\Rightarrow \tan\theta = 0.5$
Use your calculator to find the angle	Since you are finding the *angle*,	use $\tan^{-1}$ on your calculator	Don't forget to set your calculator to `deg' mode!	**Equation:** $\Rightarrow \theta = 26.6^{\circ}$

The following videos provide a summary of what you have learnt so far.
Trigonometry - 1
[missing_resource: http://www.youtube.com/v/F21S9Wpi0y8&rel=0]
Khan academy video on trigonometry - 2
[missing_resource: http://www.youtube.com/v/QS4r_mqs-rY&rel=0]

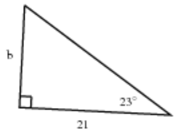
Finding Lengths

Find the length of the sides marked with letters. Give answers correct to 2 decimal places.

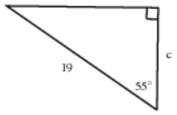
a)



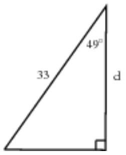
b)



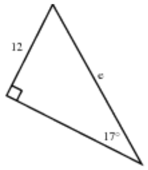
c)



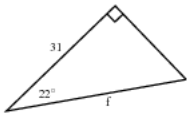
d)



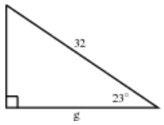
e)



f)



g)



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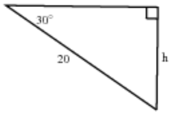
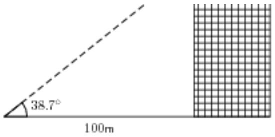


[Click here for the solution.]

Simple Applications of Trigonometric Functions

Trigonometry was probably invented in ancient civilisations to solve practical problems such as building construction and navigating by the stars. In this section we will show how trigonometry can be used to solve some other practical problems.

Height and Depth



Determining the height of a building using trigonometry.

One simple task is to find the height of a building by using trigonometry. We could just use a tape measure lowered from the roof, but this is impractical (and dangerous) for tall buildings. It is much more sensible to measure a distance along the ground and use trigonometry to find the height of the building.

[link] shows a building whose height we do not know. We have walked 100 m away from the building and measured the angle from the ground up to the top of the building. This angle is found to be $38,7°$. We call this angle the *angle of elevation*. As you can see from [link], we now have a right-angled triangle. As we know the length of one side and an angle, we can calculate the height of the triangle, which is the height of the building we are trying to find.

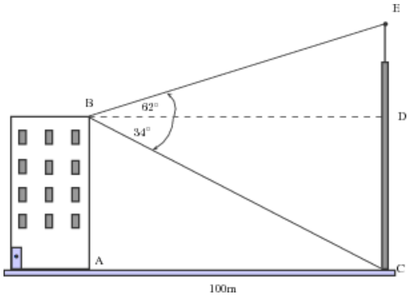
If we examine the figure, we see that we have the *opposite* and the *adjacent* of the angle of elevation and we can write:
Equation:

$$\begin{aligned}\tan 38,7° &= \frac{\text{opposite}}{\text{adjacent}}\\ &= \frac{\text{height}}{100\,\text{m}}\\ \Rightarrow \text{height} &= 100\,\text{m}\times\ \tan 38,7°\\ &= 80\,\text{m}\end{aligned}$$

Exercise:
Height of a tower

Problem:

A block of flats is 100m away from a cellphone tower. Someone stands at B. They measure the angle from B up to the top of the tower E to be 62 °. This is the angle of elevation. They then measure the angle from B down to the bottom of the tower at C to be 34 °. This is the angle of depression.What is the height of the cellph one tower correct to 1 decimal place?



Solution:

Identify a strategy

To find the height of the tower, all we have to do is find the length of CD and DE. We see that $\triangle BDE$ and $\triangle BDC$ are both right-angled triangles. For each of the triangles, we have an angle and we have the length AD. Thus we can calculate the sides of the triangles.

Calculate

We are given that the length AC is 100m. $CABD$ is a rectangle so

$$BD = AC = 100\text{m}.$$

Equation:

$$\begin{aligned} \tan\left(C\widehat{B}D\right) &= \frac{CD}{BD} \\ \Rightarrow \quad CD &= BD\times\tan\left(C\widehat{B}D\right) \\ &= 100\times\tan 34^\circ \end{aligned}$$

Use your calculator to find that $\tan 34^\circ = 0,6745$. Using this, we find that $CD = 67,45\text{m}$

Calculate **Equation:**

$$\begin{aligned} \tan\left(D\widehat{B}E\right) &= \frac{DE}{BD} \\ \Rightarrow \quad DE &= BD\times\tan\left(D\widehat{B}E\right) \\ &= 100\times\tan 62^\circ \\ &= 188,07\,\text{m} \end{aligned}$$

Combine the previous answers

We have that the height of the tower $CE = CD + DE = 67,45\,\text{m} + 188,07\,\text{m} = 255.5\,\text{m}.$

Maps and Plans

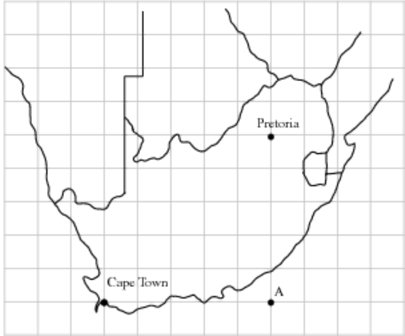
Maps and plans are usually scale drawings. This means that they are an exact copy of the real thing, but are usually smaller. So, only lengths are changed, but all angles are the same. We can use this idea to make use of maps and plans by adding information from the real world.

Exercise:

Scale Drawing

Problem:

A ship approaching Cape Town Harbour reaches point A on the map, due south of Pretoria and due east of Cape Town. If the distance from Cape Town to Pretoria is 1000km, use trigonometry to find out how far east the ship is from Cape Town, and hence find the scale of the map.



Solution:

Identify what happens in the question We already know the distance between Cape Town and A in blocks from the given map (it is 5 blocks). Thus if we work out how many kilometers this same distance is, we can calculate how many kilometers each block represents, and thus we have the scale of the map.

Identify given information Let us denote Cape Town with C and Pretoria with P. We can see that triangle APC is a right-angled triangle. Furthermore, we see that the distance AC and distance AP are both 5 blocks. Thus it is an isoceles triangle, and so $A\widehat{C}P = A\widehat{P}C = 45°$.

Carry out the calculation **Equation:**

$$\begin{aligned} CA = &\ CP\times \cos\left(A\widehat{C}P\right) \\ = &\ 1000\times \cos\left(45°\right) \\ = &\ \frac{1000}{\sqrt{2}}\ \text{km} \end{aligned}$$

To work out the scale, we see that

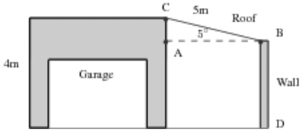
Equation:

$$\begin{aligned} & 5\ \text{blocks} = \frac{1000}{\sqrt{2}}\ \text{km} \\ \Rightarrow\ & 1\ \text{block} = \frac{200}{\sqrt{2}}\ \text{km} \end{aligned}$$

Exercise:
Building plan

Problem:

Mr Nkosi has a garage at his house, and he decides that he wants to add a corrugated iron roof to the side of the garage. The garage is 4m high, and his sheet for the roof is 5m long. If he wants the roof to be at an angle of $5°$, how high must he build the wall BD, which is holding up the roof? Give the answer to 2 decimal places.



Solution:

Set out strategy We see that the triangle ABC is a right-angled triangle. As we have one side and an angle of this triangle, we can calculate AC. The height of the wall is then the height of the garage minus AC.

Execute strategy If BC=5m, and angle $A\widehat{B}C = 5°$, then

Equation:

Thus we have that the height of the wall $BD = 4\,\text{m} - 0.4358\,\text{m} = 3.56\,\text{m}$.

$$\begin{aligned} AC &= BC\times \text{ sin } \left(A\hat{B}C\right) \\ &= 5\times \text{ sin } 5^{\circ} \\ &= 5 \times 0,0871 \\ &= 0.4358\,\text{m} \end{aligned}$$

Applications of Trigonometric Functions

1. A boy flying a kite is standing 30 m from a point directly under the kite. If the string to the kite is 50 m long, find the angle of elevation of the kite.
   Click here for the solution.
2. What is the angle of elevation of the sun when a tree 7,15 m tall casts a shadow 10,1 m long?
   Click here for the solution.

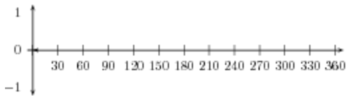
Graphs of Trigonometric Functions

This section describes the graphs of trigonometric functions.

Graph of sin θ

Graph of sin θ

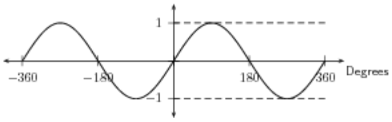
Complete the following table, using your calculator to calculate the values. Then plot the values with sin θ on the y-axis and θ on the x-axis. Round answers to 1 decimal place.

θ	0°	30°	60°	90°	120°	150°	
sin θ							
θ	180°	210°	240°	270°	300°	330°	360°
sin θ							
							

Let us look back at our values for sin θ

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$
$\sin\,\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0

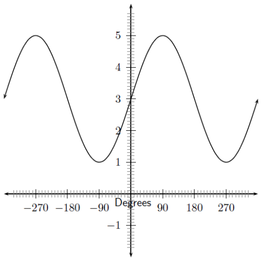
As you can see, the function $\sin\,\theta$ has a value of 0 at $\theta=0°$. Its value then smoothly increases until $\theta=90°$ when its value is 1. We also know that it later decreases to 0 when $\theta=180°$. Putting all this together we can start to picture the full extent of the sine graph. The sine graph is shown in [link]. Notice the wave shape, with each wave having a length of $360°$. We say the graph has a *period* of $360°$. The height of the wave above (or below) the x-axis is called the wave's *amplitude*. Thus the maximum amplitude of the sine-wave is 1, and its minimum amplitude is -1.



The graph of $\sin\,\theta$.

Functions of the form $y=a\,\sin\,(x)+q$

In the equation, $y=a\,\sin\,(x)+q$, a and q are constants and have different effects on the graph of the function. The general shape of the graph of functions of this form is shown in [link] for the function $f(\theta)=2\,\sin\,\theta+3$.



Graph of $f(\theta)=2\,\sin\,\theta+3$

Functions of the Form $y=a\,\sin\,(\theta)+q$:

1. On the same set of axes, plot the following graphs:
    1. $a(\theta)=\sin\,\theta-2$
    2. $b(\theta)=\sin\,\theta-1$
    3. $c(\theta)=\sin\,\theta$
    4. $d(\theta)=\sin\,\theta+1$

    5. $e(\theta) = \sin\,\theta + 2$

   Use your results to deduce the effect of q.
2. On the same set of axes, plot the following graphs:
    1. $f(\theta) = -2\cdot\,\sin\,\theta$
    2. $g(\theta) = -1\cdot\,\sin\,\theta$
    3. $h(\theta) = 0\cdot\,\sin\,\theta$
    4. $j(\theta) = 1\cdot\,\sin\,\theta$
    5. $k(\theta) = 2\cdot\,\sin\,\theta$

   Use your results to deduce the effect of a.

You should have found that the value of a affects the height of the peaks of the graph. As the magnitude of a increases, the peaks get higher. As it decreases, the peaks get lower.

q is called the *vertical shift*. If $q = 2$, then the whole sine graph shifts up 2 units. If $q = -1$, the whole sine graph shifts down 1 unit.

These different properties are summarised in [link].

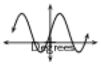
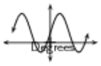
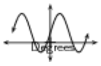
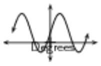
	$a > 0$	$a < 0$
$q > 0$		
$q < 0$		

Table summarising general shapes and positions of graphs of functions of the form $y = a\,\sin\,(x) + q$.

Domain and Range

For $f(\theta) = a\,\sin\,(\theta) + q$, the domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value of $\theta \in \mathbb{R}$ for which $f(\theta)$ is undefined.

The range of $f(\theta) = a\,\sin\,\theta + q$ depends on whether the value for a is positive or negative. We will consider these two cases separately.

If $a > 0$ we have:
Equation:

$$\begin{array}{rcl} -1 \leq \sin\,\theta & \leq & 1 \\ -a \leq a\,\sin\,\theta & \leq & a \quad \text{(Multiplication by a positive number maintains the nature of the inequa} \\ -a + q \leq a\,\sin\,\theta + q & \leq & a + q \\ -a + q \leq f(\theta) & \leq & a + q \end{array}$$

This tells us that for all values of θ, $f(\theta)$ is always between $-a + q$ and $a + q$. Therefore if $a > 0$, the range of $f(\theta) = a\,\sin\,\theta + q$ is $\{f(\theta) : f(\theta) \in [-a + q, a + q]\}$.

Similarly, it can be shown that if $a < 0$, the range of $f(\theta) = a \sin \theta + q$ is $\{f(\theta) : f(\theta) \in [a+q, -a+q]\}$. This is left as an exercise.

Note: The easiest way to find the range is simply to look for the "bottom" and the "top" of the graph.

Intercepts

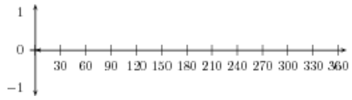
The y-intercept, y_{int}, of $f(\theta) = a \sin (x) + q$ is simply the value of $f(\theta)$ at $\theta = 0°$.
Equation:

$$\begin{aligned} y_{int} &= f(0°) \\ &= a \sin (0°) + q \\ &= a(0) + q \\ &= q \end{aligned}$$

Graph of $\cos \theta$

Graph of $\cos \theta$:

Complete the following table, using your calculator to calculate the values correct to 1 decimal place. Then plot the values with $\cos \theta$ on the y-axis and θ on the x-axis.

θ	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	
$\cos \theta$							
θ	$180°$	$210°$	$240°$	$270°$	$300°$	$330°$	$360°$
$\cos \theta$							
							

Let us look back at our values for $\cos \theta$

θ	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$

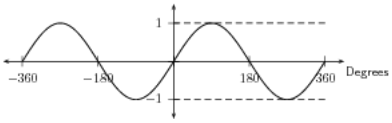
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1

If you look carefully, you will notice that the cosine of an angle θ is the same as the sine of the angle $90^\circ-\theta$. Take for example,

Equation:

$$\cos 60^\circ=\frac{1}{2}=\sin 30^\circ=\sin(90^\circ-60^\circ)$$

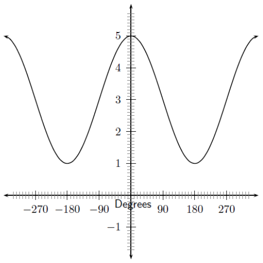
This tells us that in order to create the cosine graph, all we need to do is to shift the sine graph 90° to the left. The graph of $\cos\theta$ is shown in [link]. As the cosine graph is simply a shifted sine graph, it will have the same period and amplitude as the sine graph.



The graph of $\cos\theta$.

Functions of the form $y=a\cos(x)+q$

In the equation, $y=a\cos(x)+q$, a and q are constants and have different effects on the graph of the function. The general shape of the graph of functions of this form is shown in [link] for the function $f(\theta)=2\cos\theta+3$.



Graph of $f(\theta)=2\cos\theta+3$

Functions of the Form $y=a\cos(\theta)+q$:

1. On the same set of axes, plot the following graphs:
    1. $a(\theta)=\cos\theta-2$
    2. $b(\theta)=\cos\theta-1$

    3. $c(\theta) = \cos\theta$
    4. $d(\theta) = \cos\theta + 1$
    5. $e(\theta) = \cos\theta + 2$

   Use your results to deduce the effect of q.
2. On the same set of axes, plot the following graphs:

    1. $f(\theta) = -2 \cdot \cos\theta$
    2. $g(\theta) = -1 \cdot \cos\theta$
    3. $h(\theta) = 0 \cdot \cos\theta$
    4. $j(\theta) = 1 \cdot \cos\theta$
    5. $k(\theta) = 2 \cdot \cos\theta$

   Use your results to deduce the effect of a.

You should have found that the value of a affects the amplitude of the cosine graph in the same way it did for the sine graph.

You should have also found that the value of q shifts the cosine graph in the same way as it did the sine graph.

These different properties are summarised in [link].

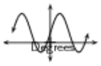
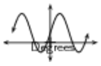
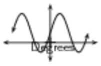
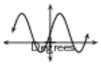
	$a > 0$	$a < 0$
$q > 0$		
$q < 0$		

Table summarising general shapes and positions of graphs of functions of the form $y = a\cos(x) + q$.

Domain and Range

For $f(\theta) = a\cos(\theta) + q$, the domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value of $\theta \in \mathbb{R}$ for which $f(\theta)$ is undefined.

It is easy to see that the range of $f(\theta)$ will be the same as the range of $a\sin(\theta) + q$. This is because the maximum and minimum values of $a\cos(\theta) + q$ will be the same as the maximum and minimum values of $a\sin(\theta) + q$.

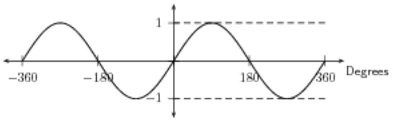
Intercepts

The y-intercept of $f(\theta) = a\cos(x) + q$ is calculated in the same way as for sine.

Equation:

$$\begin{aligned} y_{int} &= f(0^\circ) \\ &= a\cos(0^\circ) + q \\ &= a(1) + q \\ &= a + q \end{aligned}$$

Comparison of Graphs of $\sin\theta$ and $\cos\theta$



The graph of $\cos\theta$ (solid-line) and the graph of $\sin\theta$ (dashed-line).

Notice that the two graphs look very similar. Both oscillate up and down around the x-axis as you move along the axis. The distances between the peaks of the two graphs is the same and is constant along each graph. The height of the peaks and the depths of the troughs are the same.

The only difference is that the $\sin$ graph is shifted a little to the right of the $\cos$ graph by 90°. That means that if you shift the whole $\cos$ graph to the right by 90° it will overlap perfectly with the $\sin$ graph. You could also move the $\sin$ graph by $90^\circ$to the left and it would overlap perfectly with the $\cos$ graph. This means that:

Equation:

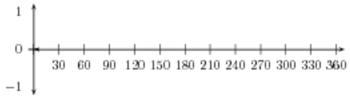
$$\begin{aligned} \sin\theta &= \cos(\theta - 90) \quad \text{(shift the cos graph to the right)} \\ &\textbf{a}\text{nd} \\ \cos\theta &= \sin(\theta + 90) \quad \text{(shift the sin graph to the left)} \end{aligned}$$

Graph of $\tan\theta$

Graph of $\tan\theta$

Complete the following table, using your calculator to calculate the values correct to 1 decimal place. Then plot the values with $\tan\theta$ on the y-axis and θ on the x-axis.

θ	0°	30°	60°	90°	120°	150°	
$\tan\theta$							

θ	180°	210°	240°	270°	300°	330°	360°
$\tan\theta$							
							

Let us look back at our values for $\tan\theta$

θ	0°	30°	45°	60°	90°	180°
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0

Now that we have graphs for $\sin\theta$ and $\cos\theta$, there is an easy way to visualise the tangent graph. Let us look back at our definitions of $\sin\theta$ and $\cos\theta$ for a right-angled triangle.

Equation:

$$\frac{\sin\theta}{\cos\theta}=\frac{\frac{\text{opposite}}{\text{hypotenuse}}}{\frac{\text{adjacent}}{\text{hypotenuse}}}=\frac{\text{opposite}}{\text{adjacent}}=\tan\theta$$

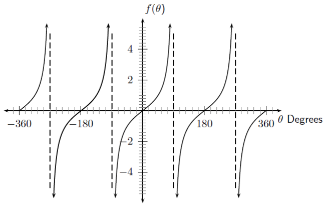
This is the first of an important set of equations called *trigonometric identities*. An identity is an equation, which holds true for any value which is put into it. In this case we have shown that

Equation:

$$\tan\theta=\frac{\sin\theta}{\cos\theta}$$

for any value of θ.

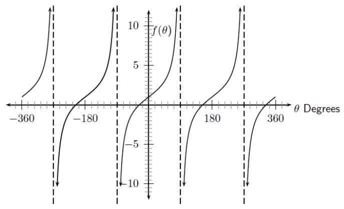
So we know that for values of θ for which $\sin\theta=0$, we must also have $\tan\theta=0$. Also, if $\cos\theta=0$ our value of $\tan\theta$ is undefined as we cannot divide by 0. The graph is shown in [link]. The dashed vertical lines are at the values of θ where $\tan\theta$ is not defined.



The graph of $\tan\theta$.

Functions of the form $y = a\ \tan\ (x) + q$

In the figure below is an example of a function of the form $y = a\ \tan\ (x) + q$.



The graph of $2\ \tan\ \theta + 1$.

Functions of the Form $y = a\ \tan\ (\theta) + q$:

1. On the same set of axes, plot the following graphs:
    1. $a(\theta)$ =tan $\theta - 2$
    2. $b(\theta)$ =tan $\theta - 1$
    3. $c(\theta)$ =tan θ
    4. $d(\theta)$ =tan $\theta + 1$
    5. $e(\theta)$ =tan $\theta + 2$

   Use your results to deduce the effect of q.
2. On the same set of axes, plot the following graphs:
    1. $f(\theta) = -2\cdot\ \tan\ \theta$
    2. $g(\theta) = -1\cdot\ \tan\ \theta$
    3. $h(\theta) = 0\cdot\ \tan\ \theta$
    4. $j(\theta) = 1\cdot\ \tan\ \theta$
    5. $k(\theta) = 2\cdot\ \tan\ \theta$

   Use your results to deduce the effect of a.

You should have found that the value of a affects the steepness of each of the branches. The larger the absolute magnitude of a, the quicker the branches approach their asymptotes, the values where they are not defined. Negative a values switch the direction of the branches. You should have also found that the value of q affects the vertical shift as for sin θ and cos θ. These different properties are summarised in [link].

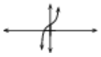
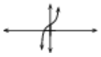
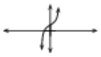
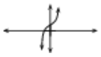
	$a > 0$	$a < 0$
$q > 0$		
$q < 0$		

Table summarising general shapes and positions of graphs of functions of the form $y = a \tan(x) + q$.

Domain and Range

The domain of $f(\theta) = a \tan(\theta) + q$ is all the values of θ such that $\cos \theta$ is not equal to 0. We have already seen that when $\cos \theta = 0$, $\tan \theta = \frac{\sin\theta}{\cos\theta}$ is undefined, as we have division by zero. We know that $\cos \theta = 0$ for all $\theta = 90^\circ + 180^\circ n$, where n is an integer. So the domain of $f(\theta) = a \tan(\theta) + q$ is all values of θ, except the values $\theta = 90^\circ + 180^\circ n$.

The range of $f(\theta) = a \tan \theta + q$ is $\{f(\theta) : f(\theta) \in (-\infty, \infty)\}$.

Intercepts

The y-intercept, y_{int}, of $f(\theta) = a \tan(x) + q$ is again simply the value of $f(\theta)$ at $\theta = 0^\circ$.
Equation:

$$\begin{aligned} y_{int} &= f(0^\circ) \\ &= a \tan(0^\circ) + q \\ &= a(0) + q \\ &= q \end{aligned}$$

Asymptotes

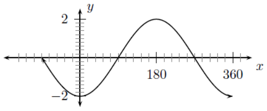
As θ approaches 90°, $\tan \theta$ approaches infinity. But as θ is undefined at 90°, θ can only approach 90°, but never equal it. Thus the $\tan \theta$ curve gets closer and closer to the line $\theta = 90^\circ$, without ever touching it. Thus the line $\theta = 90^\circ$ is an asymptote of $\tan \theta$. $\tan \theta$ also has asymptotes at $\theta = 90^\circ + 180^\circ n$, where n is an integer.

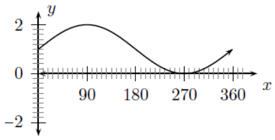
Graphs of Trigonometric Functions

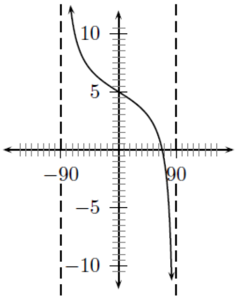
1. Using your knowldge of the effects of a and q, sketch each of the following graphs, without using a table of values, for $\theta \in [0^\circ; 360^\circ]$

   a. $y = 2 \sin \theta$
   b. $y = -4 \cos \theta$
   c. $y = -2 \cos \theta + 1$
   d. $y = \sin \theta - 3$
   e. $y = \tan \theta - 2$
   f. $y = 2 \cos \theta - 1$

   Click here for the solution.
2. Give the equations of each of the following graphs:







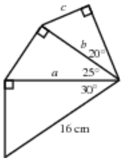
Click here for the solution.

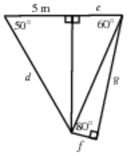
The following presentation summarises what you have learnt in this chapter. Ignore the last slide.

[missing_resource: http://static.slidesharecdn.com/swf/ssplayer2.swf?doc=p0008-wynberggirlshigh-louisekeegan-maths-grade10-trigonometry-100930084905-phpapp01&stripped_title=wynberg-girls-highlouise-keeganmathsgrade10trigonometry]

End of Chapter Exercises

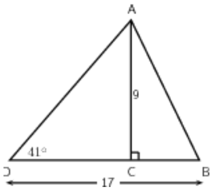
1. Calculate the unknown lengths

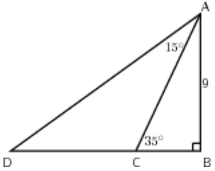
   Click here for the solution.
2. In the triangle PQR, $PR = 20$ cm, $QR = 22$ cm and $P\widehat{R}Q = 30°$. The perpendicular line from P to QR intersects QR at X. Calculate
   a. the length XR,
   b. the length PX, and
   c. the angle $Q\widehat{P}X$

   Click here for the solution.
3. A ladder of length 15 m is resting against a wall, the base of the ladder is 5 m from the wall. Find the angle between the wall and the ladder?
   Click here for the solution.
4. A ladder of length 25 m is resting against a wall, the ladder makes an angle $37°$ to the wall. Find the distance between the wall and the base of the ladder?
   Click here for the solution.
5. In the following triangle find the angle $A\widehat{B}C$



Click here for the solution.

6. In the following triangle find the length of side CD

   Click here for the solution.
7. $A(5;0)$ and $B(11;4)$. Find the angle between the line through A and B and the x-axis.
   Click here for the solution.
8. $C(0;-13)$ and $D(-12;14)$. Find the angle between the line through C and D and the y-axis.
   Click here for the solution.
9. A $5\,\text{m}$ ladder is placed $2\,\text{m}$ from the wall. What is the angle the ladder makes with the wall?
   Click here for the solution.
10. Given the points: E(5;0), F(6;2) and G(8;-2), find angle $F\widehat{E}G$.
    Click here for the solution.
11. An isosceles triangle has sides 9 cm, 9 cm and 2 cm. Find the size of the smallest angle of the triangle.
    Click here for the solution.
12. A right-angled triangle has hypotenuse 13 mm. Find the length of the other two sides if one of the angles of the triangle is 50°.
    Click here for the solution.
13. One of the angles of a rhombus (**rhombus** - A four-sided polygon, each of whose sides is of equal length) with perimeter 20 cm is 30°.

    a. Find the sides of the rhombus.
    b. Find the length of both diagonals.

    Click here for the solution.
14. Captain Hook was sailing towards a lighthouse with a height of $10\,\text{m}$.

    a. If the top of the lighthouse is $30\,\text{m}$ away, what is the angle of elevation of the boat to the nearest integer?
    b. If the boat moves another $7\,\text{m}$ towards the lighthouse, what is the new angle of elevation of the boat to the nearest integer?

    Click here for the solution.
15. (Tricky) A triangle with angles 40°, 40° and 100° has a perimeter of 20 cm. Find the length of each side of the triangle.
    Click here for the solution.

Trigonometry - Grade 11

History of Trigonometry

Work in pairs or groups and investigate the history of the development of trigonometry. Describe the various stages of development and how different cultures used trigonometry to improve their lives.

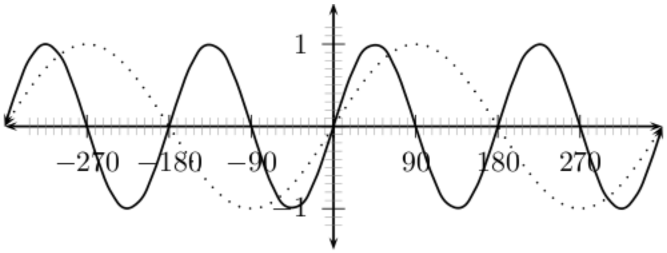
The works of the following people or cultures can be investigated:

1. Cultures
    1. Ancient Egyptians
    2. Mesopotamians
    3. Ancient Indians of the Indus Valley
2. People
    1. Lagadha (circa 1350-1200 BC)
    2. Hipparchus (circa 150 BC)
    3. Ptolemy (circa 100)
    4. Aryabhata (circa 499)
    5. Omar Khayyam (1048-1131)
    6. Bhaskara (circa 1150)
    7. Nasir al-Din (13th century)
    8. al-Kashi and Ulugh Beg (14th century)
    9. Bartholemaeus Pitiscus (1595)

Graphs of Trigonometric Functions

Functions of the form $y = \sin(k\theta)$

In the equation, $y = \sin(k\theta)$, k is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in [link] for the function $f(\theta) = \sin(2\theta)$.



Graph of $f(\theta) = \sin(2\theta)$ (solid line) and the graph of $g(\theta) = \sin(\theta)$ (dotted line).

On the same set of axes, plot the following graphs:

1. $a(\theta) = \sin 0,5\theta$
2. $b(\theta) = \sin 1\theta$
3. $c(\theta) = \sin 1,5\theta$
4. $d(\theta) = \sin 2\theta$
5. $e(\theta) = \sin 2,5\theta$

Use your results to deduce the effect of k.

You should have found that the value of k affects the period or frequency of the graph. Notice that in the case of the sine graph, the period (length of one wave) is given by $\frac{360°}{k}$.

These different properties are summarised in [link].

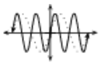
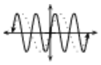
$k > 0$	$k < 0$
	

Table summarising general shapes and positions of graphs of functions of the form $y = \sin(kx)$. The curve $y = \sin(x)$ is shown as a dotted line.

Domain and Range

For $f(\theta) = \sin(k\theta)$, the domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value of $\theta \in \mathbb{R}$ for which $f(\theta)$ is undefined.

The range of $f(\theta) = \sin(k\theta)$ is $\{f(\theta) : f(\theta) \in [-1, 1]\}$.

Intercepts

For functions of the form, $y = \sin(k\theta)$, the details of calculating the intercepts with the y axis are given.

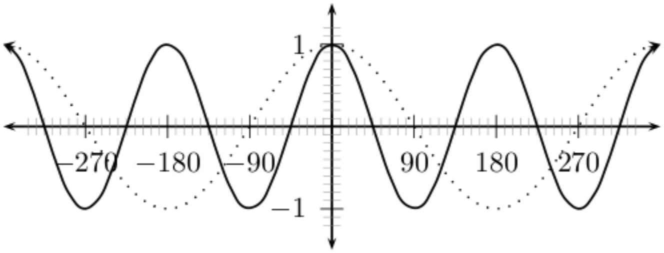
There are many x-intercepts.

The y-intercept is calculated by setting $\theta = 0$:
Equation:

$$\begin{aligned} y &= \sin(k\theta) \\ y_{int} &= \sin(0) \\ &= 0 \end{aligned}$$

Functions of the form $y = \cos(k\theta)$

In the equation, $y = \cos(k\theta)$, k is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in [link] for the function $f(\theta) = \cos(2\theta)$.



Graph of $f(\theta) = \cos(2\theta)$ (solid line) and the graph of $g(\theta) = cos(\theta)$ (dotted line).

Functions of the form $y = \cos(k\theta)$

On the same set of axes, plot the following graphs:

1. $a(\theta) = \cos 0,5\theta$
2. $b(\theta) = \cos 1\theta$
3. $c(\theta) = \cos 1,5\theta$
4. $d(\theta) = \cos 2\theta$
5. $e(\theta) = \cos 2,5\theta$

Use your results to deduce the effect of k.

You should have found that the value of k affects the period or frequency of the graph. The period of the cosine graph is given by $\frac{360°}{k}$.

These different properties are summarised in [link].

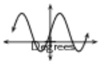
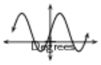
$k > 0$	$k < 0$
	

Table summarising general shapes and positions of graphs of functions of the form $y = \cos(kx)$. The curve $y = \cos(x)$ is plotted with a dotted line.

Domain and Range

For $f(\theta) = \cos(k\theta)$, the domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value of $\theta \in \mathbb{R}$ for which $f(\theta)$ is undefined.

The range of $f(\theta) = \cos(k\theta)$ is $\{f(\theta) : f(\theta) \in [-1, 1]\}$.

Intercepts

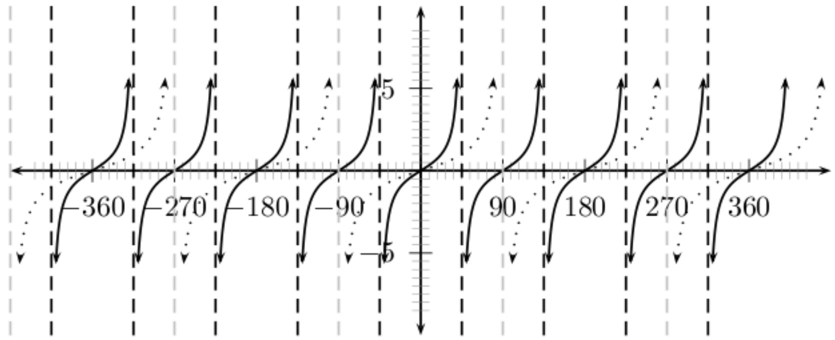
For functions of the form, $y = \cos(k\theta)$, the details of calculating the intercepts with the y axis are given.

The y-intercept is calculated as follows:
Equation:

$$\begin{aligned} y &= \cos(k\theta) \\ y_{int} &= \cos(0) \\ &= 1 \end{aligned}$$

Functions of the form $y = \tan(k\theta)$

In the equation, $y = \tan(k\theta)$, k is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in [link] for the function $f(\theta) = \tan(2\theta)$.



The graph of $\tan(2\theta)$ (solid line) and the graph of $g(\theta) = \tan(\theta)$ (dotted line). The asymptotes are shown as dashed lines.

Functions of the form $y = \tan(k\theta)$

On the same set of axes, plot the following graphs:

1. $a(\theta) = \tan 0,5\theta$
2. $b(\theta) = \tan 1\theta$
3. $c(\theta) = \tan 1,5\theta$
4. $d(\theta) = \tan 2\theta$
5. $e(\theta) = \tan 2,5\theta$

Use your results to deduce the effect of k.

You should have found that, once again, the value of k affects the periodicity (i.e. frequency) of the graph. As k increases, the graph is more tightly packed. As k decreases, the graph is more spread out. The period of the tan graph is given by $\frac{180^\circ}{k}$.

These different properties are summarised in [link].

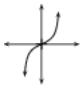
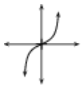
$k > 0$	$k < 0$
	

Table summarising general shapes and positions of graphs of functions of the form $y = \tan\left(k\theta\right)$.

Domain and Range

For $f(\theta) = \tan\left(k\theta\right)$, the domain of one branch is $\left\{\theta : \theta \in \left(-\frac{90^\circ}{k}, \frac{90^\circ}{k}\right)\right\}$ because the function is undefined for $\theta = -\frac{90^\circ}{k}$ and $\theta = \frac{90^\circ}{k}$.

The range of $f(\theta) = \tan\left(k\theta\right)$ is $\{f(\theta) : f(\theta) \in (-\infty, \infty)\}$.

Intercepts

For functions of the form, $y = \tan\left(k\theta\right)$, the details of calculating the intercepts with the x and y axis are given.

There are many x-intercepts; each one is halfway between the asymptotes.

The y-intercept is calculated as follows:
Equation:

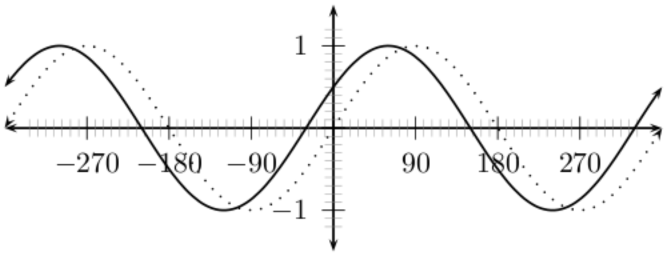
$$\begin{array}{rcl} y & = & \tan\left(k\theta\right) \\ y_{int} & = & \tan\left(0\right) \\ & = & 0 \end{array}$$

Asymptotes

The graph of $\tan k\theta$ has asymptotes because as $k\theta$ approaches 90°, $\tan k\theta$ approaches infinity. In other words, there is no defined value of the function at the asymptote values.

Functions of the form $y = \sin\left(\theta + p\right)$

In the equation, $y = \sin\left(\theta + p\right)$, p is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in [link] for the function $f\left(\theta\right) = \sin\left(\theta + 30^\circ\right)$.



Graph of $f\left(\theta\right) = \sin\left(\theta + 30°\right)$ (solid line) and the graph of $g(\theta) = \sin\left(\theta\right)$ (dotted line).

Functions of the Form $y = \sin\left(\theta + p\right)$

On the same set of axes, plot the following graphs:

1. $a\left(\theta\right) = \sin\left(\theta - 90°\right)$
2. $b\left(\theta\right) = \sin\left(\theta - 60°\right)$
3. $c(\theta) = \sin\theta$
4. $d\left(\theta\right) = \sin\left(\theta + 90°\right)$
5. $e\left(\theta\right) = \sin\left(\theta + 180°\right)$

Use your results to deduce the effect of p.

You should have found that the value of p affects the position of the graph along the y-axis (i.e. the y-intercept) and the position of the graph along the x-axis (i.e. the *phase shift*). The p value shifts the graph horizontally. If p is positive, the graph shifts left and if p is negative tha graph shifts right.

These different properties are summarised in [link].

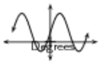
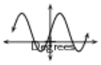
$p > 0$	$p < 0$
	

Table summarising general shapes and positions of graphs of functions of the form $y = \sin\left(\theta + p\right)$. The curve $y = \sin\left(\theta\right)$ is plotted with a dotted line.

Domain and Range

For $f(\theta) = \sin\left(\theta + p\right)$, the domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value of $\theta \in \mathbb{R}$ for which $f(\theta)$ is undefined.

The range of $f(\theta) = \sin\left(\theta + p\right)$ is $\{f(\theta) : f(\theta) \in [-1, 1]\}$.

Intercepts

For functions of the form, $y = \sin\left(\theta + p\right)$, the details of calculating the intercept with the y axis are given.

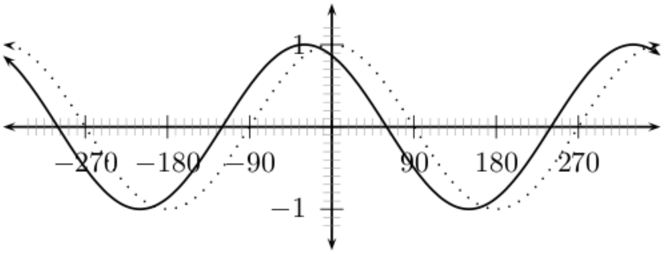
The y-intercept is calculated as follows: set $\theta = 0°$

Equation:

$$\begin{array}{rcl} y & = & \sin\left(\theta + p\right) \\ y_{int} & = & \sin\left(0 + p\right) \\ & = & \sin\left(p\right) \end{array}$$

Functions of the form $y = \cos(\theta + p)$

In the equation, $y = \cos(\theta + p)$, p is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in [link] for the function $f(\theta) = \cos(\theta + 30°)$.



Graph of $f(\theta) = \cos(\theta + 30°)$ (solid line) and the graph of $g(\theta) = \cos(\theta)$ (dotted line).

Functions of the Form $y = \cos(\theta + p)$

On the same set of axes, plot the following graphs:

1. $a(\theta) = \cos(\theta - 90°)$
2. $b(\theta) = \cos(\theta - 60°)$
3. $c(\theta) = \cos\theta$
4. $d(\theta) = \cos(\theta + 90°)$
5. $e(\theta) = \cos(\theta + 180°)$

Use your results to deduce the effect of p.

You should have found that the value of p affects the y-intercept and phase shift of the graph. As in the case of the sine graph, positive values of p shift the cosine graph left while negative p values shift the graph right.

These different properties are summarised in [link].

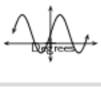
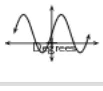
$p > 0$	$p < 0$
	

Table summarising general shapes and positions of graphs of functions of the form $y = \cos(\theta + p)$. The curve $y = \cos\theta$ is plotted with a dotted line.

Domain and Range

For $f(\theta) = \cos(\theta + p)$, the domain is $\{\theta : \theta \in \mathbb{R}\}$ because there is no value of $\theta \in \mathbb{R}$ for which $f(\theta)$ is undefined.

The range of $f(\theta) = \cos(\theta + p)$ is $\{f(\theta) : f(\theta) \in [-1, 1]\}$.

Intercepts

For functions of the form, $y = \cos(\theta + p)$, the details of calculating the intercept with the y axis are given.

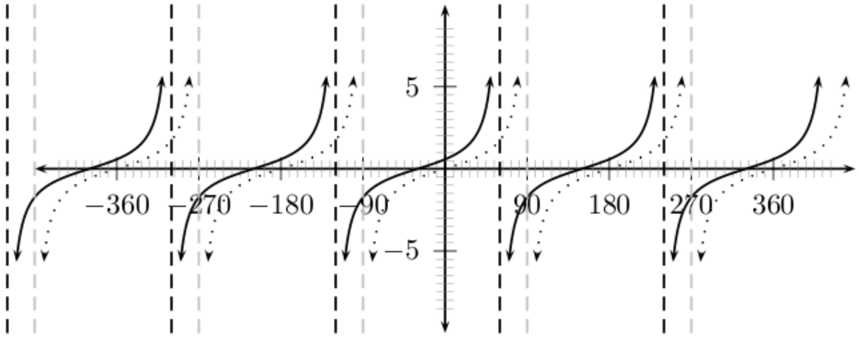
The y-intercept is calculated as follows: set $\theta = 0°$

Equation:

$$\begin{array}{rcl} y & = & \cos(\theta + p) \\ y_{int} & = & \cos(0 + p) \\ & = & \cos(p) \end{array}$$

Functions of the form $y = \tan(\theta + p)$

In the equation, $y = \tan(\theta + p)$, p is a constant and has different effects on the graph of the function. The general shape of the graph of functions of this form is shown in [link] for the function $f(\theta) = \tan(\theta + 30°)$.



The graph of $\tan(\theta + 30°)$ (solid lines) and the graph of $g(\theta) = \tan(\theta)$ (dotted lines).

Functions of the Form $y = \tan(\theta + p)$

On the same set of axes, plot the following graphs:

1. $a(\theta) = \tan(\theta - 90°)$
2. $b(\theta) = \tan(\theta - 60°)$
3. $c(\theta) = \tan\theta$
4. $d(\theta) = \tan(\theta + 60°)$
5. $e(\theta) = \tan(\theta + 180°)$

Use your results to deduce the effect of p.

You should have found that the value of p once again affects the y-intercept and phase shift of the graph. There is a horizontal shift to the left if p is positive and to the right if p is negative.

These different properties are summarised in [link].

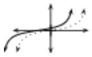
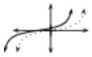
$k > 0$	$k < 0$
	

Table summarising general shapes and positions of graphs of functions of the form $y = \tan(\theta + p)$. The curve $y = \tan(\theta)$ is plotted with a dotted line.

Domain and Range

For $f(\theta) = \tan(\theta + p)$, the domain for one branch is $\{\theta : \theta \in (-90^\circ - p, 90^\circ - p\}$ because the function is undefined for $\theta = -90^\circ - p$ and $\theta = 90^\circ - p$.

The range of $f(\theta) = \tan(\theta + p)$ is $\{f(\theta) : f(\theta) \in (-\infty, \infty)\}$.

Intercepts

For functions of the form, $y = \tan(\theta + p)$, the details of calculating the intercepts with the y axis are given.

The y-intercept is calculated as follows: set $\theta = 0^\circ$
Equation:

$$\begin{aligned} y &= \tan(\theta + p) \\ y_{int} &= \tan(p) \end{aligned}$$

Asymptotes

The graph of $\tan(\theta + p)$ has asymptotes because as $\theta + p$ approaches 90°, $\tan(\theta + p)$ approaches infinity. Thus, there is no defined value of the function at the asymptote values.

Functions of various form

Using your knowledge of the effects of p and k draw a rough sketch of the following graphs without a table of values.

1. $y = \sin 3x$
2. $y = -\cos 2x$
3. $y = \tan \frac{1}{2}x$
4. $y = \sin(x - 45^\circ)$

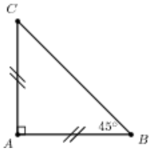
5. $y = \cos\left(x + 45^\circ\right)$
6. $y = \tan\left(x - 45^\circ\right)$
7. $y = 2\sin 2x$
8. $y = \sin\left(x + 30^\circ\right) + 1$

Trigonometric Identities

Deriving Values of Trigonometric Functions for 30°, 45° and 60°

Keeping in mind that trigonometric functions apply only to right-angled triangles, we can derive values of trigonometric functions for 30°, 45° and 60°. We shall start with 45° as this is the easiest.

Take any right-angled triangle with one angle 45°. Then, because one angle is 90°, the third angle is also 45°. So we have an isosceles right-angled triangle as shown in [link].



An isosceles right angled triangle.

If the two equal sides are of length a, then the hypotenuse, h, can be calculated as:
Equation:

$$\begin{aligned} h^2 &= a^2 + a^2 \\ &= 2a^2 \\ \therefore \quad h &= \sqrt{2}a \end{aligned}$$

So, we have:
Equation:

$$\begin{aligned} \sin\left(45^\circ\right) &= \frac{\text{opposite}(45^\circ)}{\text{hypotenuse}} \\ &= \frac{a}{\sqrt{2}a} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

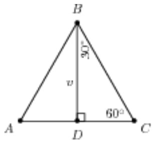
Equation:

$$\begin{aligned}\cos\left(45^\circ\right) &= \frac{\text{adjacent}(45^\circ)}{\text{hypotenuse}} \\ &= \frac{a}{\sqrt{2}a} \\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

Equation:

$$\begin{aligned}\tan\left(45^\circ\right) &= \frac{\text{opposite}(45^\circ)}{\text{adjacent}(45^\circ)} \\ &= \frac{a}{a} \\ &= 1\end{aligned}$$

We can try something similar for 30° and 60°. We start with an equilateral triangle and we bisect one angle as shown in [link]. This gives us the right-angled triangle that we need, with one angle of 30° and one angle of 60°.



An equilateral triangle with one angle bisected.

If the equal sides are of length a, then the base is $\frac{1}{2}a$ and the length of the vertical side, v, can be calculated as:
Equation:

$$\begin{aligned}v^2 &= a^2 - \left(\frac{1}{2}a\right)^2 \\ &= a^2 - \frac{1}{4}a^2 \\ &= \frac{3}{4}a^2 \\ \therefore \quad v &= \frac{\sqrt{3}}{2}a\end{aligned}$$

So, we have:
Equation:

$$\begin{aligned}\sin\left(30^\circ\right) &= \frac{\text{opposite}(30^\circ)}{\text{hypotenuse}}\\ &= \frac{\frac{a}{2}}{a}\\ &= \frac{1}{2}\end{aligned}$$

Equation:

$$\begin{aligned}\cos\left(30^\circ\right) &= \frac{\text{adjacent}(30^\circ)}{\text{hypotenuse}}\\ &= \frac{\frac{\sqrt{3}}{2}a}{a}\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

Equation:

$$\begin{aligned}\tan\left(30^\circ\right) &= \frac{\text{opposite}(30^\circ)}{\text{adjacent}(30^\circ)}\\ &= \frac{\frac{a}{2}}{\frac{\sqrt{3}}{2}a}\\ &= \frac{1}{\sqrt{3}}\end{aligned}$$

Equation:

$$\begin{aligned}\sin\left(60^\circ\right) &= \frac{\text{opposite}(60^\circ)}{\text{hypotenuse}}\\ &= \frac{\frac{\sqrt{3}}{2}a}{a}\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

Equation:

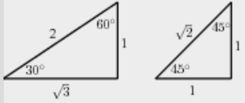
$$\begin{aligned}\cos\left(60^\circ\right) &= \frac{\text{adjacent}(60^\circ)}{\text{hypotenuse}}\\ &= \frac{\frac{a}{2}}{a}\\ &= \frac{1}{2}\end{aligned}$$

Equation:

$$\begin{aligned}\tan\left(60^\circ\right) &= \frac{\text{opposite}(60^\circ)}{\text{adjacent}(60^\circ)}\\ &= \frac{\frac{\sqrt{3}}{2}a}{\frac{a}{2}}\\ &= \sqrt{3}\end{aligned}$$

You do not have to memorise these identities if you know how to work them out.

Note:
Two useful triangles to remember



Alternate Definition for $\tan\theta$

We know that $\tan\theta$ is defined as: $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$ This can be written as:
Equation:

$$\begin{aligned}\tan\theta &= \frac{\text{opposite}}{\text{adjacent}} \times \frac{\text{hypotenuse}}{\text{hypotenuse}} \\ &= \frac{\text{opposite}}{\text{hypotenuse}} \times \frac{\text{hypotenuse}}{\text{adjacent}}\end{aligned}$$

But, we also know that $\sin\theta$ is defined as: $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ and that $\cos\theta$ is defined as: $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$

Therefore, we can write
Equation:

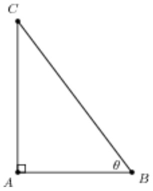
$$\begin{aligned}\tan\theta &= \frac{\text{opposite}}{\text{hypotenuse}} \times \frac{\text{hypotenuse}}{\text{adjacent}} \\ &= \sin\theta \times \frac{1}{\cos\theta} \\ &= \frac{\sin\theta}{\cos\theta}\end{aligned}$$

Note:$\tan\theta$ can also be defined as: $\tan\theta = \frac{\sin\theta}{\cos\theta}$

A Trigonometric Identity

One of the most useful results of the trigonometric functions is that they are related to each other. We have seen that $\tan\theta$ can be written in terms of $\sin\theta$ and $\cos\theta$. Similarly, we shall show that: $\sin^2\theta + \cos^2\theta = 1$

We shall start by considering $\triangle ABC$,



We see that: $\sin\theta = \frac{AC}{BC}$ and $\cos\theta = \frac{AB}{BC}$.

We also know from the Theorem of Pythagoras that: $AB^2 + AC^2 = BC^2$.

So we can write:
Equation:

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= \left(\frac{AC}{BC}\right)^2 + \left(\frac{AB}{BC}\right)^2 \\ &= \frac{AC^2}{BC^2} + \frac{AB^2}{BC^2} \\ &= \frac{AC^2 + AB^2}{BC^2} \\ &= \frac{BC^2}{BC^2} \qquad (\text{from Pythagoras}) \\ &= 1\end{aligned}$$

Exercise:
Trigonometric Identities A

Problem: Simplify using identities:

1. $\tan^2\theta \cdot \cos^2\theta$
2. $\frac{1}{\cos^2\theta} - \tan^2\theta$

Solution:

Use known identities to Equation: replace

$$\begin{aligned} &= \tan^2\theta \cdot \cos^2\theta \\ &= \frac{\sin^2\theta}{\cos^2\theta} \cdot \cos^2\theta \\ &= \sin^2\theta \end{aligned}$$

Use known identities to Equation: replace

$$\begin{aligned} &= \frac{1}{\cos^2\theta} - \tan^2\theta \\ &= \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta} \\ &= \frac{1 - \sin^2\theta}{\cos^2\theta} \\ &= \frac{\cos^2\theta}{\cos^2\theta} = 1 \end{aligned}$$

Exercise:
Trigonometric Identities B

Problem: Prove: $\frac{1-\sin x}{\cos x} = \frac{\cos x}{1+\sin x}$

Solution:

Use trig identities **Equation:**

$$\begin{aligned} \text{LHS} &= \frac{1-\sin x}{\cos x} \\ &= \frac{1-\sin x}{\cos x} \times \frac{1+\sin x}{1+\sin x} \\ &= \frac{1-\sin^2 x}{\cos x(1+\sin x)} \\ &= \frac{\cos^2 x}{\cos x(1+\sin x)} \\ &= \frac{\cos x}{1+\sin x} = \text{RHS} \end{aligned}$$

Trigonometric identities

1. Simplify the following using the fundamental trigonometric identities:
    1. $\frac{\cos\theta}{\tan\theta}$
    2. $\cos^2\theta.\tan^2\theta + \tan^2\theta.\sin^2\theta$
    3. $1-\tan^2\theta.\sin^2\theta$
    4. $1-\sin\theta.\cos\theta.\tan\theta$
    5. $1-\sin^2\theta$
    6. $\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)-\cos^2\theta$
2. Prove the following:
    1. $\frac{1+\sin\theta}{\cos\theta} = \frac{\cos\theta}{1-\sin\theta}$
    2. $\sin^2\theta + (\cos\theta - \tan\theta)(\cos\theta + \tan\theta) = 1-\tan^2\theta$
    3. $\frac{(2\cos^2\theta-1)}{1} + \frac{1}{(1+\tan^2\theta)} = \frac{1-\tan^2\theta}{1+\tan^2\theta}$
    4. $\frac{1}{\cos\theta} - \frac{\cos\theta\tan^2\theta}{1} = 1$
    5. $\frac{2\sin\theta\cos\theta}{\sin\theta+\cos\theta} = \sin\theta + \cos\theta - \frac{1}{\sin\theta+\cos\theta}$
    6. $\left(\frac{\cos\theta}{\sin\theta} + \tan\theta\right)\cdot\cos\theta = \frac{1}{\sin\theta}$

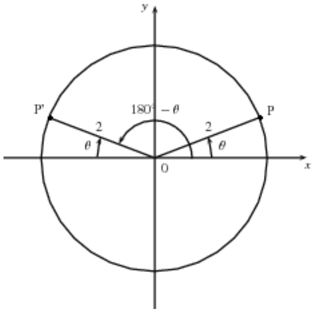
Reduction Formula

Any trigonometric function whose argument is $90^\circ \pm \theta$, $180^\circ \pm \theta$, $270^\circ \pm \theta$ and $360^\circ \pm \theta$ (hence $-\theta$) can be written simply in terms of θ. For example, you may have noticed that the cosine graph is identical to the sine graph except for a phase shift of 90°. From this we may expect that $\sin(90^\circ+\theta) = \cos\theta$.

Function Values of $180^\circ \pm \theta$

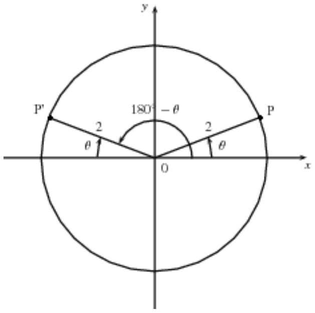
Investigation : Reduction Formulae for Function Values of $180^\circ \pm \theta$

1. **Function Values of** $(180^\circ - \theta)$
    1. In the figure P and P' lie on the circle with radius 2. OP makes an angle $\theta = 30^\circ$ with the x-axis. P thus has coordinates $\left(\sqrt{3}; 1\right)$. If P' is the reflection of P about the y-axis (or the line $x = 0$), use symmetry to write down the coordinates of P'.
    2. Write down values for $\sin\theta$, $\cos\theta$ and $\tan\theta$.
    3. Using the coordinates for P' determine $\sin(180^\circ - \theta)$, $\cos(180^\circ - \theta)$ and $\tan(180^\circ - \theta)$.



    1. **(d)**From your results try and determine a relationship between the function values of $(180^\circ - \theta)$ and θ.

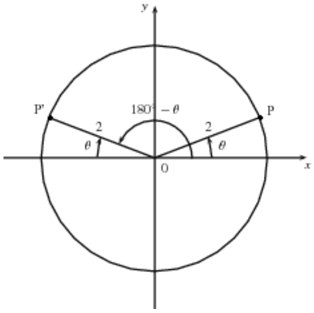
2. **Function values of** $(180^\circ + \theta)$
    1. In the figure P and P' lie on the circle with radius 2. OP makes an angle $\theta = 30^\circ$ with the x-axis. P thus has coordinates $\left(\sqrt{3}; 1\right)$. P' is the inversion of P through the origin (reflection about both the x- and y-axes) and lies at an angle of $180^\circ + \theta$ with the x-axis. Write down the coordinates of P'.
    2. Using the coordinates for P' determine $\sin(180^\circ + \theta)$, $\cos(180^\circ + \theta)$ and $\tan(180^\circ + \theta)$.
    3. From your results try and determine a relationship between the function values of $(180^\circ + \theta)$ and θ.



Investigation : Reduction Formulae for Function Values of $360^\circ \pm \theta$

1. **Function values of** $(360^\circ - \theta)$
    1. In the figure P and P' lie on the circle with radius 2. OP makes an angle $\theta = 30^\circ$ with the x-axis. P thus has coordinates $\left(\sqrt{3}; 1\right)$. P' is the reflection of P about the x-axis or the line $y = 0$. Using symmetry, write down the coordinates of P'.
    2. Using the coordinates for P' determine $\sin(360^\circ - \theta)$, $\cos(360^\circ - \theta)$ and $\tan(360^\circ - \theta)$.

3. From your results try and determine a relationship between the function values of $(360^\circ - \theta)$ and θ.



It is possible to have an angle which is larger than 360°. The angle completes one revolution to give 360° and then continues to give the required angle. We get the following results:
Equation:

$$\begin{aligned} \sin(360^\circ + \theta) &= \sin\theta \\ \cos(360^\circ + \theta) &= \cos\theta \\ \tan(360^\circ + \theta) &= \tan\theta \end{aligned}$$

Note also, that if k is any integer, then
Equation:

$$\begin{aligned} \sin(k360^\circ + \theta) &= \sin\theta \\ \cos(k360^\circ + \theta) &= \cos\theta \\ \tan(k360^\circ + \theta) &= \tan\theta \end{aligned}$$

Exercise:
Basic use of a reduction formula

Problem: Write $\sin 293^\circ$ as the function of an acute angle.

Solution:

We note that thus **Equation:**

$$\begin{aligned} \sin 293^\circ &= \sin(360^\circ - 67^\circ) \\ &= -\sin 67^\circ \end{aligned}$$

where we used the fact that $\sin(360^\circ - \theta) = -\sin\theta$. Check, using your calculator, that these values are in fact equal:

Equation:

$$\begin{aligned} \sin 293^\circ &= -0,92\cdots \\ -\sin 67^\circ &= -0,92\cdots \end{aligned}$$

Exercise:
More complicated...

Problem: Evaluate without using a calculator:
Equation:

$$\tan^2 210^\circ - (1 + \cos 120^\circ)\sin^2 225^\circ$$

Solution:

Rewrite each angle as acute angles **Equation:**

$$\begin{aligned} & \tan^2 210^\circ - (1+ \cos 120^\circ) \sin^2 225^\circ \\ = \; & [\tan (180^\circ + 30^\circ)]^2 - [1+ \cos (180^\circ - 60^\circ)] \cdot [\sin (180^\circ + 45^\circ)]^2 \\ = \; & (\tan 30^\circ)^2 - [1 + (-\cos 60^\circ)] \cdot (-\sin 45^\circ)^2 \\ = \; & \left(\frac{1}{\sqrt{3}}\right)^2 - \left(1 - \frac{1}{2}\right) \cdot \left(-\frac{1}{\sqrt{2}}\right)^2 \\ = \; & \frac{1}{3} - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \\ = \; & \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \end{aligned}$$

Reduction Formulae

1. Write these equations as a function of θ only:
    1. $\sin (180^\circ - \theta)$
    2. $\cos (180^\circ - \theta)$
    3. $\cos (360^\circ - \theta)$
    4. $\cos (360^\circ + \theta)$
    5. $\tan (180^\circ - \theta)$
    6. $\cos (360^\circ + \theta)$
2. Write the following trig functions as a function of an acute angle:
    1. $\sin 163^\circ$
    2. $\cos 327^\circ$
    3. $\tan 248^\circ$
    4. $\cos 213^\circ$
3. Determine the following without the use of a calculator:
    1. $\tan \; 150^\circ . \sin \; 30^\circ + \cos \; 330^\circ$
    2. $\tan \; 300^\circ . \cos \; 120^\circ$
    3. $(1- \cos \; 30^\circ)(1- \sin \; 210^\circ)$
    4. $\cos \; 780^\circ + \sin \; 315^\circ . \tan \; 420^\circ$
4. Determine the following by reducing to an acute angle and using special angles. Do not use a calculator:
    1. $\cos \; 300^\circ$
    2. $\sin \; 135^\circ$
    3. $\cos \; 150^\circ$
    4. $\tan \; 330^\circ$
    5. $\sin \; 120^\circ$
    6. $\tan^2 225^\circ$
    7. $\cos \; 315^\circ$
    8. $\sin^2 420^\circ$
    9. $\tan \; 405^\circ$
    10. $\cos \; 1020^\circ$
    11. $\tan^2 135^\circ$
    12. $1- \sin^2 210^\circ$

Function Values of $(-\theta)$

When the argument of a trigonometric function is $(-\theta)$ we can add $360°$ without changing the result. Thus for sine and cosine

Equation:

$$\sin\,(-\theta) = \sin\,(360° - \theta) = -\,\sin\theta$$

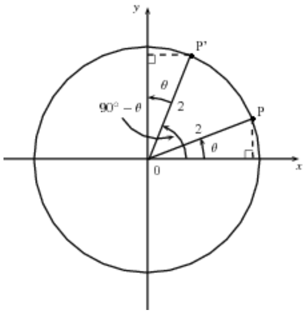
Equation:

$$\cos\,(-\theta) = \cos\,(360° - \theta) = \cos\theta$$

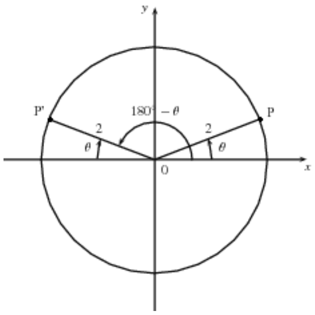
Function Values of $90° \pm \theta$

Investigation : Reduction Formulae for Function Values of $90° \pm \theta$

1. **Function values of** $(90° - \theta)$
    1. In the figure P and P' lie on the circle with radius 2. OP makes an angle $\theta = 30°$ with the x-axis. P thus has coordinates $\left(\sqrt{3};1\right)$. P' is the reflection of P about the line $y = x$. Using symmetry, write down the coordinates of P'.
    2. Using the coordinates for P' determine $\sin\,(90° - \theta)$, $\cos\,(90° - \theta)$ and $\tan\,(90° - \theta)$.
    3. From your results try and determine a relationship between the function values of $(90° - \theta)$ and θ.



2. **Function values of** $(90° + \theta)$
    1. In the figure P and P' lie on the circle with radius 2. OP makes an angle $\theta = 30°$ with the x-axis. P thus has coordinates $\left(\sqrt{3};1\right)$. P' is the rotation of P through $90°$. Using symmetry, write down the coordinates of P'. (Hint: consider P' as the reflection of P about the line $y = x$ followed by a reflection about the y-axis)
    2. Using the coordinates for P' determine $\sin\,(90° + \theta)$, $\cos\,(90° + \theta)$ and $\tan\,(90° + \theta)$.
    3. From your results try and determine a relationship between the function values of $(90° + \theta)$ and θ.



Complementary angles are positive acute angles that add up to $90°$. e.g. $20°$ and $70°$ are complementary angles.

Sine and cosine are known as **co-functions**. Two functions are called co-functions if $f(A) = g(B)$ whenever $A + B = 90°$ (i.e. A and B are complementary angles). The other trig co-functions are secant and cosecant, and tangent and cotangent.

The function value of an angle is equal to the co-function of its complement (the co-co rule).

Thus for sine and cosine we have
Equation:

$$\begin{aligned}\sin(90° - \theta) &= \cos\theta \\ \cos(90° - \theta) &= \sin\theta\end{aligned}$$

Exercise:
Co-co rule

Problem:

Write each of the following in terms of $40°$ using $\sin(90° - \theta) = \cos\theta$ and $\cos(90° - \theta) = \sin\theta$.

1. $\cos 50°$
2. $\sin 320°$
3. $\cos 230°$

Solution:

We use what we have learnt so far:

1. $\cos 50° = \cos(90° - 40°) = \sin 40°$
2. $\sin 320° = \sin(360° - 40°) = -\sin 40°$
3. $\cos 230° = \cos(180° + 50°) = -\cos 50° = -\cos(90° - 40°) = -\sin 40°$

Function Values of $(\theta - 90°)$

$\sin(\theta - 90°) = -\cos\theta$ and $\cos(\theta - 90°) = \sin\theta$.

These results may be proved as follows:
Equation:

$$\begin{aligned} \sin(\theta - 90^\circ) &= \sin[-(90^\circ - \theta)] \\ &= -\sin(90^\circ - \theta) \\ &= -\cos\theta \end{aligned}$$

and likewise for cos $(\theta - 90^\circ)$ =sin θ

Summary

The following summary may be made

second quadrant $(180^\circ - \theta)$ **or** $(90^\circ + \theta)$	**first quadrant** (θ) **or** $(90^\circ - \theta)$
$\sin(180^\circ - \theta) = +\sin\theta$	all trig functions are positive
$\cos(180^\circ - \theta) = -\cos\theta$	$\sin(360^\circ + \theta) = \sin\theta$
$\tan(180^\circ - \theta) = -\tan\theta$	$\cos(360^\circ + \theta) = \cos\theta$
$\sin(90^\circ + \theta) = +\cos\theta$	$\tan(360^\circ + \theta) = \tan\theta$
$\cos(90^\circ + \theta) = -\sin\theta$	$\sin(90^\circ - \theta) = \sin\theta$
	$\cos(90^\circ - \theta) = \cos\theta$
third quadrant $(180^\circ + \theta)$	**fourth quadrant** $(360^\circ - \theta)$
$\sin(180^\circ + \theta) = -\sin\theta$	$\sin(360^\circ - \theta) = -\sin\theta$
$\cos(180^\circ + \theta) = -\cos\theta$	$\cos(360^\circ - \theta) = +\cos\theta$
$\tan(180^\circ + \theta) = +\tan\theta$	$\tan(360^\circ - \theta) = -\tan\theta$

Note:

1. These reduction formulae hold for any angle θ. For convenience, we usually work with θ as if it is acute, i.e. $0^\circ < \theta < 90^\circ$.
2. When determining function values of $180^\circ \pm \theta$, $360^\circ \pm \theta$ and $-\theta$ the functions never change.
3. When determining function values of $90^\circ \pm \theta$ and $\theta - 90^\circ$ the functions changes to its co-function (co-co rule).

Function Values of $(270^\circ \pm \theta)$

Angles in the third and fourth quadrants may be written as $270^\circ \pm \theta$ with θ an acute angle. Similar rules to the above apply. We get

third quadrant $(270^\circ - \theta)$	**fourth quadrant** $(270^\circ + \theta)$
$\sin\,(270^\circ - \theta) = -\,\cos\,\theta$	$\sin\,(270^\circ + \theta) = -\,\cos\,\theta$
$\cos\,(270^\circ - \theta) = -\,\sin\,\theta$	$\cos\,(270^\circ + \theta) = +\,\sin\,\theta$

Solving Trigonometric Equations

In Grade 10 and 11 we focussed on the solution of algebraic equations and excluded equations that dealt with trigonometric functions (i.e. sin and cos). In this section, the solution of trigonometric equations will be discussed.

The methods described in previous Grades also apply here. In most cases, trigonometric identities will be used to simplify equations, before finding the final solution. The final solution can be found either graphically or using inverse trigonometric functions.

Graphical Solution

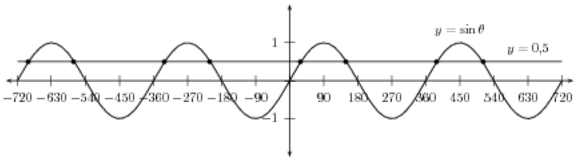
As an example, to introduce the methods of solving trigonometric equations, consider
Equation:

$$\sin\,\theta = 0,5$$

As explained in previous Grades,the solution of Equation [link] is obtained by examining the intersecting points of the graphs of:
Equation:

$$\begin{array}{rcl} y & = & \sin\,\theta \\ y & = & 0,5 \end{array}$$

Both graphs, for $-720^\circ < \theta < 720^\circ$, are shown in [link] and the intersection points of the graphs are shown by the dots.



Plot of $y =$sin θ and $y = 0,5$ showing the points of intersection, hence the solutions to the equation sin $\theta = 0,5$.

In the domain for θ of $-720^\circ < \theta < 720^\circ$, there are 8 possible solutions for the equation sin $\theta = 0,5$. These are $\theta = [-690^\circ, -570^\circ, -330^\circ, -210^\circ, 30^\circ, 150^\circ, 390^\circ, 510^\circ]$
Exercise:

Problem: Find θ, if $\tan\,\theta + 0,5 = 1,5$, with $0^\circ < \theta < 90^\circ$. Determine the solution graphically.

Solution:

Write the equation so that all the terms with the unknown quantity (i.e.) are on one side of the equation.

Equation:

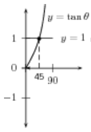
$$\begin{aligned} \tan\theta + 0,5 &= 1,5 \\ \tan\theta &= 1 \end{aligned}$$

Identify the two functions which are intersecting.

Equation:

$$\begin{aligned} y &= \tan\theta \\ y &= 1 \end{aligned}$$

Draw graphs of both functions, over the required domain and identify the intersection point.



The graphs intersect at $\theta = 45^\circ$.

Algebraic Solution

The inverse trigonometric functions $\arcsin$, $\arccos$ and $\arctan$ can also be used to solve trigonometric equations. These may be shown as second functions on your calculator: $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$.

Using inverse trigonometric functions, the equation

Equation:

$$\sin\theta = 0,5$$

is solved as

Equation:

$$\begin{aligned} \sin\theta &= 0,5 \\ \therefore \quad \theta &= \arcsin 0,5 \\ &= 30^\circ \end{aligned}$$

Exercise:

Problem:

Find θ, if $\tan\theta + 0,5 = 1,5$, with $0^\circ < \theta < 90^\circ$. Determine the solution using inverse trigonometric functions.

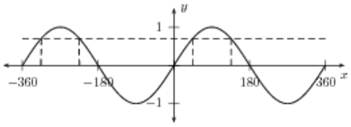
Solution:

Write the equation so that all the terms with the unknown quantity (i.e.) are on one side of the equation. Then solve for the angle using the inverse function.

Equation:

$$\begin{aligned} \tan\theta + 0,5 &= 1,5 \\ \tan\theta &= 1 \\ \therefore \quad \theta &= \arctan 1 \\ &= 45^\circ \end{aligned}$$

Trigonometric equations often look very simple. Consider solving the equation $\sin\theta = 0,7$. We can take the inverse sine of both sides to find that $\theta = \sin^{-1}(0,7)$. If we put this into a calculator we find that $\sin^{-1}(0,7) = 44,42°$. This is true, however, it does not tell the whole story.



The sine graph. The dotted line represents $y = 0,7$. There are four points of intersection on this interval, thus four solutions to $\sin\theta = 0,7$.

As you can see from [link], there are **four** possible angles with a sine of $0,7$ between $-360°$ and $360°$. If we were to extend the range of the sine graph to infinity we would in fact see that there are an infinite number of solutions to this equation! This difficulty (which is caused by the periodicity of the sine function) makes solving trigonometric equations much harder than they may seem to be.

Any problem on trigonometric equations will require two pieces of information to solve. The first is the equation itself and the second is the **range** in which your answers must lie. The hard part is making sure you find all of the possible answers within the range. Your calculator will always give you the **smallest** answer (*i.e.* the one that lies between $-90°$ and $90°$ for tangent and sine and one between $0°$ and $180°$ for cosine). Bearing this in mind we can already solve trigonometric equations within these ranges.

Exercise:

Problem: Find the values of x for which $\sin\left(\frac{x}{2}\right) = 0,5$ if it is given that $x < 90°$.

Solution:

Determine if the angle is acute Because we are told that x is an acute angle, we can simply apply an inverse trigonometric function to both sides.

Equation:

$$\begin{aligned} \sin\left(\frac{x}{2}\right) &= 0,5 \\ \Rightarrow \frac{x}{2} &= \arcsin 0,5 \\ \Rightarrow \frac{x}{2} &= 30° \\ \therefore x &= 60° \end{aligned}$$

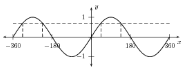
We can, of course, solve trigonometric equations in any range by drawing the graph.

Exercise:

Problem: For what values of x does $\sin x = 0,5$, when $-360° \leq x \leq 360°$?

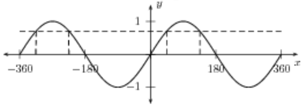
Solution:

Draw the graph We take a look at the graph of $\sin x = 0,5$ on the interval $[-360°, 360°]$. We want to know when the y value of the graph is $0,5$, so we draw in a line at $y = 0,5$.



Notice that this line touches the graph four times. This means that there are four solutions to the equation.

Read off the x values of those intercepts from the graph as $x = -330°$, $-210°$, $30°$ and $150°$.

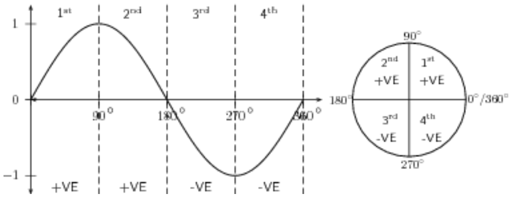


This method can be time consuming and inexact. We shall now look at how to solve these problems algebraically.

Solution using CAST diagrams

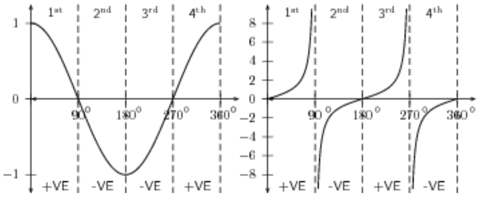
The Sign of the Trigonometric Function

The first step to finding the trigonometry of any angle is to determine the **sign** of the ratio for a given angle. We shall do this for the sine function first and then do the same for the cosine and tangent.



The graph and unit circle showing the sign of the sine function.

In [link] we have split the sine graph into four **quadrants**, each $90°$ wide. We call them quadrants because they correspond to the four quadrants of the unit circle. We notice from [link] that the sine graph is positive in the 1^{st} and 2^{nd} quadrants and negative in the 3^{rd} and 4^{th}. [link] shows similar graphs for cosine and tangent.



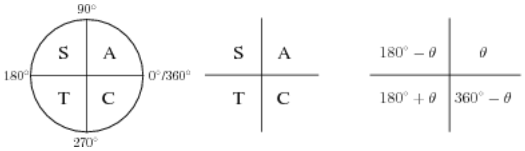
Graphs showing the sign of the cosine and tangent functions.

All of this can be summed up in two ways. [link] shows which trigonometric functions are positive and which are negative in each quadrant.

	1st	2nd	3rd	4th
sin	+VE	+VE	-VE	-VE
cos	+VE	-VE	-VE	+VE
tan	+VE	-VE	+VE	-VE

The signs of the three basic trigonometric functions in each quadrant.

A more convenient way of writing this is to note that all functions are positive in the 1st quadrant, only sine is positive in the 2nd, only tangent in the 3rd and only cosine in the 4th. We express this using the CAST diagram ([link]). This diagram is known as a CAST diagram as the letters, taken anticlockwise from the bottom right, read C-A-S-T. The letter in each quadrant tells us which trigonometric functions are **positive** in that quadrant. The `A' in the 1st quadrant stands for all (meaning sine, cosine and tangent are all positive in this quadrant). $S\prime$,C' and `T' ,of course, stand for sine, cosine and tangent. The diagram is shown in two forms. The version on the left shows the CAST diagram including the unit circle. This version is useful for equations which lie in large or negative ranges. The simpler version on the right is useful for ranges between 0° and 360°. Another useful diagram shown in [link] gives the formulae to use in each quadrant when solving a trigonometric equation.



The two forms of the CAST diagram and the formulae in each quadrant.

Magnitude of the trigonometric functions

Now that we know in which quadrants our solutions lie, we need to know which angles in these quadrants satisfy our equation.

Calculators give us the smallest possible answer (sometimes negative) which satisfies the equation. For example, if we wish to solve $\sin\theta = 0,3$ we can apply the inverse sine function to both sides of the equation to find:
Equation:

$$\begin{aligned} \theta &= \arcsin 0,3 \\ &= 17,46^\circ \end{aligned}$$

However, we know that this is just one of infinitely many possible answers. We get the rest of the answers by finding relationships between this small angle, θ, and answers in other quadrants. To do this we use our small angle θ as a **reference angle**. We then look at the sign of the trigonometric function in order to decide in which quadrants we need to work (using the CAST diagram) and add multiples of the period to each, remembering that sine, cosine and tangent are periodic (repeating) functions. To add multiples of the period we use $360^\circ \cdot n$ (where n is an integer) for sine and cosine and $180^\circ \cdot n$, $n \in \mathbb{Z}$, for the tangent.
Exercise:

Problem: Solve for θ:

Equation:

$$\sin\theta = 0,3$$

Solution:

Determine in which quadrants the solution lies

We look at the sign of the trigonometric function. $\sin\theta$ is given as a positive amount $(0,3)$. Reference to the CAST diagram shows that sine is positive in the first and second quadrants.

S	A
T	C

Determine the reference angle

The small angle θ is the angle returned by the calculator:

Equation:

$$\begin{aligned} \sin\theta &= 0,3 \\ \Rightarrow \theta &= \arcsin 0,3 \\ \Rightarrow \theta &= 17,46^\circ \end{aligned}$$

Determine the general solution

Our solution lies in quadrants I and II. We therefore use θ and $180^\circ - \theta$, and add the $360^\circ \cdot n$ for the periodicity of sine.

$180^\circ - \theta$	θ
$180^\circ + \theta$	$360^\circ - \theta$

Equation:

$$\begin{aligned} \text{I}: \quad \theta &= 17,46^\circ + 360^\circ \cdot n, n \in \mathbb{Z} \\ \text{II}: \quad \theta &= 180^\circ - 17,46^\circ + 360^\circ \cdot n, n \in \mathbb{Z} \\ &= 162,54^\circ + 360^\circ \cdot n, n \in \mathbb{Z} \end{aligned}$$

This is called the **general solution**.

Find the specific solutions

We can then find **all** the values of θ by substituting $n = ..., -1, 0, 1, 2, ...$etc. For example, If $n = 0, \quad \theta = 17,46^\circ; 162,54^\circ$ If $n = 1, \quad \theta = 377,46^\circ; 522,54^\circ$ If $n = -1, \quad \theta = -342,54^\circ; -197,46^\circ$ We can find as many as we like or find specific solutions in a given interval by choosing more values for n.

General Solution Using Periodicity

Up until now we have only solved trigonometric equations where the argument (the bit after the function, e.g. the θ in $\cos\theta$ or the $(2x-7)$ in $\tan(2x-7)$), has been θ. If there is anything more complicated than this we need to be a little more careful. Let us try to solve $\tan(2x - 10^\circ) = 2,5$ in the range $-360^\circ \leq x \leq 360^\circ$. We want solutions for positive tangent so using our CAST diagram we know to look in the 1st and 3rd quadrants. Our calculator tells us that $\arctan(2,5) = 68,2^\circ$. This is our reference angle. So to find the general solution we proceed as follows:

Equation:

$$\begin{aligned}\tan\,(2x-10^\circ) &= 2,5\ [68,2^\circ]\\ \text{I}:\quad 2\text{x}-10^\circ &= 68,2^\circ+180^\circ\cdot n\\ 2x &= 78,2^\circ+180^\circ\cdot n\\ x &= 39,1^\circ+90^\circ\cdot n, n\in\mathbb{Z}\end{aligned}$$

This is the general solution. Notice that we added the 10° and divided by 2 only at the end. Notice that we added $180^\circ\cdot n$ because the tangent has a period of 180°. This is **also** divided by 2 in the last step to keep the equation balanced. We chose quadrants I and III because $\tan$ was positive and we used the formulae θ in quadrant I and $(180^\circ+\theta)$ in quadrant III. To find solutions where $-360^\circ<x<360^\circ$ we substitue integers for n:

- $n=0$; $x=39,1^\circ$; $129,1^\circ$
- $n=1$; $x=129,1^\circ$; $219,1^\circ$
- $n=2$; $x=219,1^\circ$; $309,1^\circ$
- $n=3$; $x=309,1^\circ$; $399,1^\circ$ (too big!)
- $n=-1$; $x=-50,9^\circ$; $39,1^\circ$
- $n=-2$; $x=-140,1^\circ$; $-50,9^\circ$
- $n=-3$; $x=-230,9^\circ$; $-140,9^\circ$
- $n=-4$; $x=-320,9^\circ$; $-230,9^\circ$

Solution: $x=-320,9^\circ;-230^\circ;-140,9^\circ;-50,9^\circ;39,1^\circ;129,1^\circ;219,1^\circ$ and $309,1^\circ$

Linear Trigonometric Equations

Just like with regular equations without trigonometric functions, solving trigonometric equations can become a lot more complicated. You should solve these just like normal equations to isolate a single trigonometric ratio. Then you follow the strategy outlined in the previous section.

Exercise:

Problem: Write down the general solution for $3\cos\,(\theta-15^\circ)-1=-2,583$

Solution:

Use what we have learnt so far:

Equation:

$$\begin{aligned}3\cos\,(\theta-15^\circ)-1 &= -2,583\\ 3\cos\,(\theta-15^\circ) &= -1,583\\ \cos\,(\theta-15^\circ) &= -0,5276\ldots\\ \text{reference angle}:(\theta-15^\circ) &= 58,2^\circ\\ \text{II}:\quad \theta-15^\circ &= 180^\circ-58,2^\circ+360^\circ\cdot n, n\in\mathbb{Z}\\ \theta &= 136,8^\circ+360^\circ\cdot n, n\in\mathbb{Z}\\ \text{III}:\quad \theta-15^\circ &= 180^\circ+58,2^\circ+360^\circ\cdot n, n\in\mathbb{Z}\\ \theta &= 253,2^\circ+360^\circ\cdot n, n\in\mathbb{Z}\end{aligned}$$

Quadratic and Higher Order Trigonometric Equations

The simplest quadratic trigonometric equation is of the form

Equation:

$$\sin^2 x - 2 = -1,5$$

This type of equation can be easily solved by rearranging to get a more familiar linear equation

Equation:

$$\begin{aligned} \sin^2 x &= 0,5 \\ \Rightarrow \sin x &= \pm\sqrt{0,5} \end{aligned}$$

This gives two linear trigonometric equations. The solutions to either of these equations will satisfy the original quadratic.

The next level of complexity comes when we need to solve a trinomial which contains trigonometric functions. It is much easier in this case to use **temporary variables**. Consider solving

Equation:

$$\tan^2 (2x+1) + 3 \tan (2x+1) + 2 = 0$$

Here we notice that $\tan (2x+1)$ occurs twice in the equation, hence we let $y = \tan (2x+1)$ and rewrite:

Equation:

$$y^2 + 3y + 2 = 0$$

That should look rather more familiar. We can immediately write down the factorised form and the solutions:

Equation:

$$\begin{aligned} (y+1)(y+2) = 0 \\ \Rightarrow y = -1 \quad \text{OR} \quad \mathrm{y} = -2 \end{aligned}$$

Next we just substitute back for the temporary variable: $\tan (2x+1) = -1 \quad \text{or} \quad \tan (2\mathrm{x}+1) = -2$ And then we are left with two linear trigonometric equations. Be careful: sometimes one of the two solutions will be outside the **range** of the trigonometric function. In that case you need to discard that solution. For example consider the same equation with cosines instead of tangents $\cos^2 (2x+1) + 3 \cos (2x+1) + 2 = 0$ Using the same method we find that $\cos (2x+1) = -1 \quad \text{or} \quad \cos (2\mathrm{x}+1) = -2$ The second solution cannot be valid as cosine must lie between -1 and 1. We must, therefore, reject the second equation. Only solutions to the first equation will be valid.

More Complex Trigonometric Equations

Here are two examples on the level of the hardest trigonometric equations you are likely to encounter. They require using everything that you have learnt in this chapter. If you can solve these, you should be able to solve anything!

Exercise:

Problem: Solve $2 \cos^2 x - \cos x - 1 = 0$ for $x \in [-180°; 360°]$

Solution:

Use a temporary We note that $\cos x$ occurs twice in the equation. So, let $y = \cos x$. Then we have

variable $2y^2 - y - 1 = 0$ Note that with practice you may be able to leave out this step.

Solve the quadratic equation Factorising yields

Equation:

$$(2y+1)(y-1)=0$$

Equation:

$$\therefore \quad y=-0,5 \quad \text{or} \quad \text{y}=1$$

Substitute back and solve the two resulting equations We thus get

Equation:

$$\cos x=-0,5 \quad \text{or} \quad \cos \text{x}=1$$

Both equations are valid (*i.e.* lie in the range of cosine). General solution:

Equation:

$$\begin{aligned} \cos x &= -0,5 \quad [60^\circ] \\ \text{II}: \quad \text{x} &= 180^\circ - 60^\circ + 360^\circ \cdot n, n \in \mathbb{Z}\text{I; IV :} \\ &= 120^\circ + 360^\circ \cdot n, n \in \mathbb{Z} \\ \text{III}: \quad \text{x} &= 180^\circ + 60^\circ + 360^\circ \cdot n, n \in \mathbb{Z} \\ &= 240^\circ + 360^\circ \cdot n, n \in \mathbb{Z} \end{aligned}$$

Equat

c

Exercise:

Problem: Solve for x in the interval $[-360^\circ; 360^\circ]$:

Equation:

$$2\sin^2 x - \sin x \cos x = 0$$

Solution:

Factorise Factorising yields

Equation:

$$\sin x(2\sin x - \cos x) = 0$$

which gives two equations

Equation:

$$\sin x = 0$$

Equation:

$$\begin{aligned} 2\sin x &= \cos x \\ \frac{2\sin x}{\cos x} &= \frac{\cos x}{\cos x} \\ 2\tan x &= 1 \\ \tan x &= \tfrac{1}{2} \end{aligned}$$

Solve the two trigonometric equations General solution:

Equation:

$$\begin{aligned} \sin x &= 0 \quad [0^\circ] \\ \therefore \quad x &= 180^\circ \cdot n, n \in \mathbb{Z} \end{aligned}$$

Equation:

$$\begin{aligned} \tan x &= \tfrac{1}{2} \quad [26,57^\circ] \\ \text{I; III}: \quad \text{x} &= 26,57^\circ + 180^\circ \cdot n, n \in \mathbb{Z} \end{aligned}$$

Specific solution

$x = -360^\circ; -2$

Solving Trigonometric Equations

1. 1. Find the general solution of each of the following equations. Give answers to one decimal place.
   2. Find all solutions in the interval $\theta \in [-180^\circ; 360^\circ]$.
      1. sin $\theta = -0,327$
      2. cos $\theta = 0,231$
      3. tan $\theta = -1,375$
      4. sin $\theta = 2,439$
2. 1. Find the general solution of each of the following equations. Give answers to one decimal place.
   2. Find all solutions in the interval $\theta \in [0^\circ; 360^\circ]$.

    1. $\cos\theta = 0$
    2. $\sin\theta = \frac{\sqrt{3}}{2}$
    3. $2\cos\theta - \sqrt{3} = 0$
    4. $\tan\theta = -1$
    5. $5\cos\theta = -2$
    6. $3\sin\theta = -1,5$
    7. $2\cos\theta + 1,3 = 0$
    8. $0,5\tan\theta + 2,5 = 1,7$

3.
    1. Write down the general solution for x if $\tan x = -1,12$.
    2. Hence determine values of $x \in [-180°; 180°]$.

4.
    1. Write down the general solution for θ if $\sin\theta = -0,61$.
    2. Hence determine values of $\theta \in [0°; 720°]$.

5.
    1. Solve for A if $\sin(\mathrm{A} + 20°) = 0,53$
    2. Write down the values of A$\in [0°; 360°]$

6.
    1. Solve for x if $\cos(x + 30°) = 0,32$
    2. Write down the values of $x \in [-180°; 360°]$

7.
    1. Solve for θ if $\sin^2(\theta) + 0,5\sin\theta = 0$
    2. Write down the values of $\theta \in [0°; 360°]$

Sine and Cosine Identities

There are a few identities relating to the trigonometric functions that make working with triangles easier. These are:

1. the sine rule
2. the cosine rule
3. the area rule

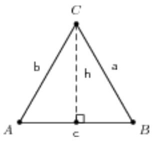
and will be described and applied in this section.

The Sine Rule

The Sine Rule

> The sine rule applies to any triangle: $\frac{\sin\hat{A}}{a} = \frac{\sin\hat{B}}{b} = \frac{\sin\hat{C}}{c}$ where a is the side opposite $\hat{A}$, b is the side opposite $\hat{B}$ and c is the side opposite $\hat{C}$.

Consider $\triangle ABC$.



The area of $\triangle ABC$ can be written as: area $\triangle\mathrm{ABC} = \frac{1}{2}\mathrm{c}\cdot\mathrm{h}$. However, h can be calculated in terms of $\hat{A}$ or $\hat{B}$ as:

Equation:

$$\begin{aligned} \sin \widehat{A} &= \frac{h}{b} \\ \therefore \quad h &= b \cdot \sin \widehat{A} \end{aligned}$$

and
Equation:

$$\begin{aligned} \sin \widehat{B} &= \frac{h}{a} \\ \therefore \quad h &= a \cdot \sin \widehat{B} \end{aligned}$$

Therefore the area of $\triangle ABC$ is: $\frac{1}{2}c \cdot h = \frac{1}{2}c \cdot b \cdot \sin \widehat{A} = \frac{1}{2}c \cdot a \cdot \sin \widehat{B}$

Similarly, by drawing the perpendicular between point B and line AC we can show that: $\frac{1}{2}c \cdot b \cdot \sin \widehat{A} = \frac{1}{2}a \cdot b \cdot \sin \widehat{C}$

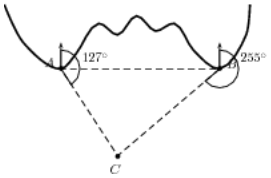
Therefore the area of $\triangle ABC$ is: $\frac{1}{2}c \cdot b \cdot \sin \widehat{A} = \frac{1}{2}c \cdot a \cdot \sin \widehat{B} = \frac{1}{2}a \cdot b \cdot \sin \widehat{C}$

If we divide through by $\frac{1}{2}a \cdot b \cdot c$, we get: $\frac{\sin\widehat{A}}{a} = \frac{\sin\widehat{B}}{b} = \frac{\sin\widehat{C}}{c}$

This is known as the sine rule and applies to *any* triangle, right angled or not.
Exercise:
Lighthouses

Problem:

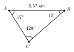
There is a coastline with two lighthouses, one on either side of a beach. The two lighthouses are $0,67$ km apart and one is exactly due east of the other. The lighthouses tell how close a boat is by taking bearings to the boat (remember – a bearing is an angle measured clockwise from north). These bearings are shown. Use the sine rule to calculate how far the boat is from each lighthouse.



Solution:

Determine what is given

We can see that the two lighthouses and the boat form a triangle. Since we know the distance between the lighthouses and we have two angles we can use trigonometry to find the remaining two sides of the triangle, the distance of the boat from the two lighthouses.



Determine what you need

We need to know the lengths of the two sides AC and BC. We can use the sine rule to find our missing lengths.

Equation:

Equation:

$$\begin{aligned} \frac{BC}{\sin\ \widehat{A}} &= \frac{AB}{\sin\ \widehat{C}} \\ BC &= \frac{AB\cdot\ \sin\ \widehat{A}}{\sin\ \widehat{C}} \\ &= \frac{(0,67\,\text{km})\ \sin\ (37^\circ)}{\sin\ (128^\circ)} \\ &= 0,51\,\text{km} \end{aligned} \qquad \begin{aligned} \frac{AC}{\sin\ \widehat{B}} &= \frac{AB}{\sin\ \widehat{C}} \\ AC &= \frac{AB\cdot\ \sin\ \widehat{B}}{\sin\ \widehat{C}} \\ &= \frac{(0,67\,\text{km})\ \sin\ (15^\circ)}{\sin\ (128^\circ)} \\ &= 0,22\,\text{km} \end{aligned}$$

Sine Rule

1. Show that $\frac{\sin\widehat{A}}{a} = \frac{\sin\widehat{B}}{b} = \frac{\sin\widehat{C}}{c}$ is equivalent to: $\frac{a}{\sin\widehat{A}} = \frac{b}{\sin\widehat{B}} = \frac{c}{\sin\widehat{C}}$ Note: either of these two forms can be used.
2. Find all the unknown sides and angles of the following triangles:
    1. $\triangle$PQR in which $\widehat{\text{Q}} = 64^\circ$; $\widehat{\text{R}} = 24^\circ$ and $r = 3$
    2. $\triangle$KLM in which $\widehat{\text{K}} = 43^\circ$; $\widehat{\text{M}} = 50^\circ$ and $m = 1$
    3. $\triangle$ABC in which $\widehat{\text{A}} = 32,7^\circ$; $\widehat{\text{C}} = 70,5^\circ$ and $a = 52,3$
    4. $\triangle$XYZ in which $\widehat{\text{X}} = 56^\circ$; $\widehat{\text{Z}} = 40^\circ$ and $x = 50$
3. In $\triangle$ABC, $\widehat{\text{A}} = 116^\circ$; $\widehat{\text{C}} = 32^\circ$ and $\text{AC} = 23$ m. Find the length of the side AB.
4. In $\triangle$RST, $\widehat{\text{R}} = 19^\circ$; $\widehat{\text{S}} = 30^\circ$ and $\text{RT} = 120$ km. Find the length of the side ST.
5. In $\triangle$KMS, $\widehat{\text{K}} = 20^\circ$; $\widehat{\text{M}} = 100^\circ$ and $s = 23$ cm. Find the length of the side m.

The Cosine Rule

The Cosine Rule

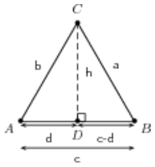
The cosine rule applies to any triangle and states that:
Equation:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\ \cos\ \widehat{A} \\ b^2 &= c^2 + a^2 - 2ca\ \cos\ \widehat{B} \\ c^2 &= a^2 + b^2 - 2ab\ \cos\ \widehat{C} \end{aligned}$$

where a is the side opposite $\widehat{A}$, b is the side opposite $\widehat{B}$ and c is the side opposite $\widehat{C}$.

The cosine rule relates the length of a side of a triangle to the angle opposite it and the lengths of the other two sides.

Consider $\triangle ABC$ which we will use to show that: $a^2 = b^2 + c^2 - 2bc\ \cos\ \widehat{A}$.



In $\triangle DCB$: $a^2 = (c-d)^2 + h^2$ from the theorem of Pythagoras.

In $\triangle ACD$: $b^2 = d^2 + h^2$ from the theorem of Pythagoras.

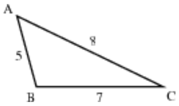
We can eliminate h^2 from [link] and [link] to get:
Equation:

$$\begin{aligned} b^2 - d^2 &= a^2 - (c-d)^2 \\ a^2 &= b^2 + \left(c^2 - 2cd + d^2\right) - d^2 \\ &= b^2 + c^2 - 2cd + d^2 - d^2 \\ &= b^2 + c^2 - 2cd \end{aligned}$$

In order to eliminate d we look at $\triangle ACD$, where we have: $\cos \widehat{A} = \frac{d}{b}$. So, $d = b \cos \widehat{A}$. Substituting this into [link], we get: $a^2 = b^2 + c^2 - 2bc \cos \widehat{A}$

The other cases can be proved in an identical manner.
Exercise:

Problem: Find $\widehat{\mathrm{A}}$:



Solution:

Applying the cosine Equation:
rule:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bd \cos \widehat{\mathrm{A}} \\ \therefore \quad \cos \widehat{\mathrm{A}} &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{8^2 + 5^2 - 7^2}{2 \cdot 8 \cdot 5} \\ &= 0,5 \\ \therefore \quad \widehat{\mathrm{A}} &= \arccos 0,5 = 60^\circ \end{aligned}$$

The Cosine Rule

1. Solve the following triangles *i.e.* find all unknown sides and angles
    1. $\triangle$ABC in which $\widehat{\mathrm{A}} = 70^\circ$; $b = 4$ and $c = 9$
    2. $\triangle$XYZ in which $\widehat{\mathrm{Y}} = 112^\circ$; $x = 2$ and $y = 3$

    3. $\triangle$RST in which RS$=2$; ST$=3$ and RT$=5$
    4. $\triangle$KLM in which KL$=5$; LM$=10$ and KM$=7$
    5. $\triangle$JHK in which $\widehat{\text{H}}=130^\circ$; JH$=13$ and HK$=8$
    6. $\triangle$DEF in which $d=4$; $e=5$ and $f=7$

2. Find the length of the third side of the $\triangle$XYZ where:

    1. $\widehat{\text{X}}=71,4^\circ$; $y=3,42$ km and $z=4,03$ km
    2. ; $x=103,2$ cm; $\widehat{\text{Y}}=20,8^\circ$ and $z=44,59$ cm

3. Determine the largest angle in:

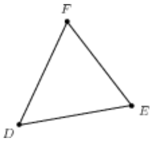
    1. $\triangle$JHK in which JH$=6$; HK$=4$ and JK$=3$
    2. $\triangle$PQR where $p=50$; $q=70$ and $r=60$

The Area Rule

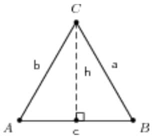
The Area Rule

> The area rule applies to any triangle and states that the area of a triangle is given by half the product of any two sides with the sine of the angle between them.

That means that in the $\triangle DEF$, the area is given by:
$A=\frac{1}{2}DE\cdot EF\sin\widehat{E}=\frac{1}{2}EF\cdot FD\sin\widehat{F}=\frac{1}{2}FD\cdot DE\sin\widehat{D}$



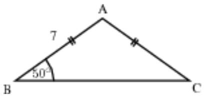
In order show that this is true for all triangles, consider $\triangle ABC$.



The area of any triangle is half the product of the base and the perpendicular height. For $\triangle ABC$, this is:
$A=\frac{1}{2}c\cdot h$. However, h can be written in terms of $\widehat{A}$ as: $h=b\sin\widehat{A}$ So, the area of $\triangle ABC$ is:
$A=\frac{1}{2}c\cdot b\sin\widehat{A}$.

Using an identical method, the area rule can be shown for the other two angles.
Exercise:
The Area Rule

Problem: Find the area of $\triangle$ABC:



Solution:

Decide what information you have

$\triangle$ABC is isosceles, therefore AB$=$AC$=7$ and $\widehat{\text{C}}=\widehat{\text{B}}=50°$. Hence $\widehat{\text{A}}=180°-50°-50°=80°$. Now we can use the area rule to find the area:

Equation:

$$\begin{aligned} A &= \frac{1}{2}cb\sin\widehat{\text{A}} \\ &= \frac{1}{2}\cdot 7\cdot 7\cdot\sin 80° \\ &= 24,13 \end{aligned}$$

The Area Rule

Draw sketches of the figures you use in this exercise.

1. Find the area of $\triangle$PQR in which:
    1. $\widehat{\text{P}}=40°$; $q=9$ and $r=25$
    2. $\widehat{\text{Q}}=30°$; $r=10$ and $p=7$
    3. $\widehat{\text{R}}=110°$; $p=8$ and $q=9$
2. Find the area of:
    1. $\triangle$XYZ with XY$=6$ cm; XZ$=7$ cm and $\widehat{\text{Z}}=28°$
    2. $\triangle$PQR with PR$=52$ cm; PQ$=29$ cm and $\widehat{\text{P}}=58,9°$
    3. $\triangle$EFG with FG$=2,5$ cm; EG$=7,9$ cm and $\widehat{\text{G}}=125°$
3. Determine the area of a parallelogram in which two adjacent sides are 10 cm and 13 cm and the angle between them is $55°$.
4. If the area of $\triangle$ABC is 5000 m^2 with $a=150$ m and $b=70$ m, what are the two possible sizes of $\widehat{\text{C}}$?

Summary of the Trigonometric Rules and Identities

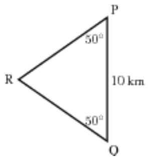
Pythagorean Identity	Ratio Identity
$\cos^2\theta+\sin^2\theta=1$	$\tan\theta=\frac{\sin\theta}{\cos\theta}$

Odd/Even Identities	Periodicity Identities	Cofunction Identities

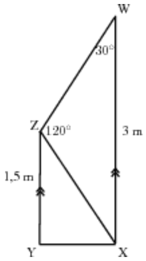
$\sin\,(-\theta) = -\,\sin\,\theta$	$\sin\,(\theta \pm 360^\circ) = \sin\,\theta$	$\sin\,(90^\circ - \theta) = \cos\,\theta$
$\cos\,(-\theta) = \cos\,\theta$	$\cos\,(\theta \pm 360^\circ) = \cos\,\theta$	$\cos\,(90^\circ - \theta) = \sin\,\theta$
Sine Rule	Area Rule	Cosine Rule
	Area $= \frac{1}{2}$bc cos A	$a^2 = b^2 + c^2 - 2bc\,\cos\,A$
$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	Area $= \frac{1}{2}$ac cos B	$b^2 = a^2 + c^2 - 2ac\,\cos\,B$
	Area $= \frac{1}{2}$ab cos C	$c^2 = a^2 + b^2 - 2ab\,\cos\,C$

Exercises

1. Q is a ship at a point 10 km due South of another ship P. R is a lighthouse on the coast such that $\widehat{\text{P}} = \widehat{\text{Q}} = 50^\circ$. Determine:
    1. the distance QR
    2. the shortest distance from the lighthouse to the line joining the two ships (PQ).

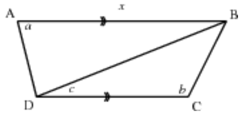


2. WXYZ is a trapezium (WX$\|$XZ) with WX$= 3$ m; YZ$= 1,5$ m;$\widehat{\text{Z}} = 120^\circ$ and $\widehat{\text{W}} = 30^\circ$
    1. Determine the distances XZ and XY.
    2. Find the angle $\widehat{\text{C}}$.



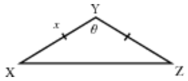
3. On a flight from Johannesburg to Cape Town, the pilot discovers that he has been flying 3° off course. At this point the plane is 500 km from Johannesburg. The direct distance between Cape Town and Johannesburg airports is 1 552 km. Determine, to the nearest km:
    1. The distance the plane has to travel to get to Cape Town and hence the extra distance that the plane has had to travel due to the pilot's error.
    2. The correction, to one hundredth of a degree, to the plane's heading (or direction).

4. ABCD is a trapezium (ie. AB$\|$CD). AB$= x$; B$\widehat{\text{A}}$D $=$ a; B$\widehat{\text{C}}$D $=$ b and B$\widehat{\text{D}}$C $=$ c. Find an expression for the length of CD in terms of x, a, b and c.

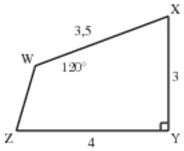


5. A surveyor is trying to determine the distance between points X and Z. However the distance cannot be determined directly as a ridge lies between the two points. From a point Y which is equidistant from X and Z, he measures the angle $\text{X}\hat{\text{Y}}\text{Z}$.

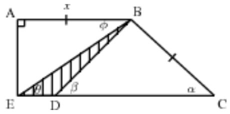
    1. If XY$=x$ and $\text{X}\hat{\text{Y}}\text{Z}=\theta$, show that XZ$=x\sqrt{2(1-\cos\theta)}$.
    2. Calculate XZ (to the nearest kilometre) if $x=240$ km and $\theta=132^\circ$.



6. Find the area of WXYZ (to two decimal places):



7. Find the area of the shaded triangle in terms of x, α, β, θ and φ:



Compound Angle Identities and Applications

Trigonometry - Grade 12

Compound Angle Identities

Derivation of $\sin\,(\alpha+\beta)$

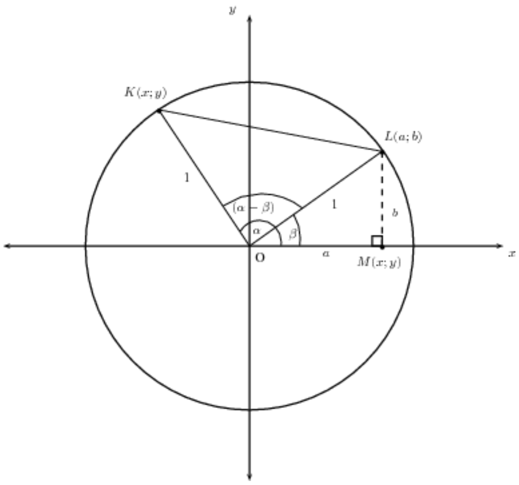
We have, for any angles α and β, that

Equation:

$$\sin\,(\alpha+\beta) = \sin\,\alpha\,\cos\,\beta + \sin\,\beta\,\cos\,\alpha$$

How do we derive this identity? It is tricky, so follow closely.

Suppose we have the unit circle shown below. The two points $L(a,b)$ and $K(x,y)$ are on the circle.



We can get the coordinates of L and K in terms of the angles α and β. For the triangle LOK, we have that

Equation:

$$\begin{array}{lcl} \sin\,\beta = \dfrac{b}{1} & \Rightarrow & b = \sin\,\beta \\ \cos\,\beta = \dfrac{a}{1} & \Rightarrow & a = \cos\,\beta \end{array}$$

Thus the coordinates of L are $(\cos\,\beta;\sin\,\beta)$. In the same way as above, we can see that the coordinates of K are $(\cos\,\alpha;\sin\,\alpha)$. The identity for $\cos\,(\alpha-\beta)$ is now determined by calculating KL^2 in two ways. Using the distance formula (i.e. $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ or $d^2=(x_2-x_1)^2+(y_2-y_1)^2$), we can find KL^2:

Equation:

$$\begin{aligned} KL^2 &= (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 \\ &= \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta \\ &= \left(\cos^2\alpha + \sin^2\alpha\right) + \left(\cos^2\beta + \sin^2\beta\right) - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta \\ &= 1 + 1 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\ &= 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \end{aligned}$$

The second way we can determine KL^2 is by using the cosine rule for $\triangle KOL$:
Equation:

$$\begin{aligned} KL^2 &= KO^2 + LO^2 - 2 \cdot KO \cdot LO \cdot \cos(\alpha - \beta) \\ &= 1^2 + 1^2 - 2(1)(1)\cos(\alpha - \beta) \\ &= 2 - 2 \cdot \cos(\alpha - \beta) \end{aligned}$$

Equating our two values for KL^2, we have
Equation:

$$\begin{aligned} 2 - 2 \cdot \cos(\alpha - \beta) &= 2 - 2(\cos\alpha\cos\beta + \sin\alpha \cdot \sin\beta) \\ \Rightarrow \quad \cos(\alpha - \beta) &= \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta \end{aligned}$$

Now let $\alpha \to 90^\circ - \alpha$. Then
Equation:

$$\begin{aligned} \cos(90^\circ - \alpha - \beta) &= \cos(90^\circ - \alpha)\cos\beta + \sin(90^\circ - \alpha)\sin\beta \\ &= \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta \end{aligned}$$

But $\cos(90^\circ - (\alpha + \beta)) = \sin(\alpha + \beta)$. Thus
Equation:

$$\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$$

Derivation of $\sin(\alpha - \beta)$

We can use
Equation:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

to show that
Equation:

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

We know that
Equation:

$$\sin(-\theta) = -\sin(\theta)$$

and

Equation:

$$\cos(-\theta) = \cos\theta$$

Therefore,

Equation:

$$\begin{aligned}\sin(\alpha-\beta) &= \sin(\alpha+(-\beta)) \\ &= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) \\ &= \sin\alpha\cos\beta - \cos\alpha\sin\beta\end{aligned}$$

Derivation of $\cos(\alpha+\beta)$

We can use

Equation:

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$

to show that

Equation:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

We know that

Equation:

$$\sin(\theta) = \cos(90-\theta).$$

Therefore,

Equation:

$$\begin{aligned}\cos(\alpha+\beta) &= \sin(90-(\alpha+\beta)) \\ &= \sin((90-\alpha)-\beta)) \\ &= \sin(90-\alpha)\cos\beta - \sin\beta\cos(90-\alpha) \\ &= \cos\alpha\cos\beta - \sin\beta\sin\alpha\end{aligned}$$

Derivation of $\cos(\alpha-\beta)$

We found this identity in our derivation of the $\sin(\alpha+\beta)$ identity. We can also use the fact that

Equation:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

to derive that

Equation:

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

As
Equation:

$$\cos(\theta)=\sin(90-\theta),$$

we have that
Equation:

$$\begin{aligned}\cos(\alpha-\beta) &= \sin(90-(\alpha-\beta))\\ &= \sin((90-\alpha)+\beta))\\ &= \sin(90-\alpha)\cos\beta+\cos(90-\alpha)\sin\beta\\ &= \cos\alpha\cos\beta+\sin\alpha\sin\beta\end{aligned}$$

Derivation of $\sin 2\alpha$

We know that
Equation:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

When $\alpha=\beta$, we have that
Equation:

$$\begin{aligned}\sin(2\alpha)=\sin(\alpha+\alpha) &= \sin\alpha\cos\alpha+\cos\alpha\sin\alpha\\ &= 2\sin\alpha\cos\alpha\\ &= \sin(2\alpha)\end{aligned}$$

Derivation of $\cos 2\alpha$

We know that
Equation:

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

When $\alpha=\beta$, we have that
Equation:

$$\begin{aligned}\cos(2\alpha)=\cos(\alpha+\alpha) &= \cos\alpha\cos\alpha-\sin\alpha\sin\alpha\\ &= \cos^2\alpha-\sin^2\alpha\\ &= \cos(2\alpha)\end{aligned}$$

However, we can also write
Equation:

$$\cos 2\alpha=2\cos^2\alpha-1$$

and
Equation:

$$\cos 2\alpha = 1 - 2\sin^2\alpha$$

by using
Equation:

$$\sin^2\alpha + \cos^2\alpha = 1.$$

The $\cos 2\alpha$ Identity

Use
Equation:

$$\sin^2\alpha + \cos^2\alpha = 1$$

to show that:
Equation:

$$\cos 2\alpha \quad = \quad \begin{cases} 2\cos^2\alpha - 1 \\ 1 - 2\sin^2\alpha \end{cases}$$

Problem-solving Strategy for Identities

The most important thing to remember when asked to prove identities is:

Note: Trigonometric Identities

When proving trigonometric identities, never assume that the left hand side is equal to the right hand side. You need to **show** that both sides are equal.

A suggestion for proving identities: It is usually much easier simplifying the more complex side of an identity to get the simpler side than the other way round.
Exercise:
Trigonometric Identities 1

Problem: Prove that $\sin 75^\circ = \frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}$ without using a calculator.

Solution:

Identify a strategy We only know the exact values of the trig functions for a few special angles (30°, 45°, 60°, etc.). We can see that $75^\circ = 30^\circ + 45^\circ$. Thus we can use our double-angle identity for $\sin\left(\alpha + \beta\right)$ to express

$\sin 75°$ in terms of known trig function values.

Execute strategy **Equation:**

$$\begin{aligned} \sin 75° &= \sin(45° + 30°) \\ &= \sin(45°)\cos(30°) + \cos(45°)\sin(30°) \\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{3}+1}{2\sqrt{2}} \\ &= \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4} \end{aligned}$$

Exercise:
Trigonometric Identities 2

Problem: Deduce a formula for $\tan(\alpha + \beta)$ in terms of $\tan\alpha$ and $\tan\beta$.

Hint: Use the formulae for $\sin(\alpha + \beta)$ *and* $\cos(\alpha + \beta)$

Solution:

Identify a strategy We can express $\tan(\alpha + \beta)$ in terms of cosines and sines, and then use the double-angle formulas for these. We then manipulate the resulting expression in order to get it in terms of $\tan\alpha$ and $\tan\beta$.

Execute strategy **Equation:**

$$\begin{aligned} \tan(\alpha + \beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\ &= \frac{\sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta}{\cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta} \\ &= \frac{\frac{\sin\alpha \cdot \cos\beta}{\cos\alpha \cdot \cos\beta} + \frac{\cos\alpha \cdot \sin\beta}{\cos\alpha \cdot \cos\beta}}{\frac{\cos\alpha \cdot \cos\beta}{\cos\alpha \cdot \cos\beta} - \frac{\sin\alpha \cdot \sin\beta}{\cos\alpha \cdot \cos\beta}} \\ &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta} \end{aligned}$$

Exercise:
Trigonometric Identities 3

Problem: Prove that
Equation:

$$\frac{\sin\theta + \sin 2\theta}{1 + \cos\theta + \cos 2\theta} = \tan\theta$$

In fact, this identity is not valid for all values of θ. Which values are those?

Solution:

Identify a strategy

The right-hand side (RHS) of the identity cannot be simplified. Thus we should try simplify the left-hand side (LHS). We can also notice that the trig function on the RHS does not have a 2θ dependance. Thus we will need to use the double-angle formulas to simplify the $\sin 2\theta$ and $\cos 2\theta$ on the LHS. We know that $\tan\theta$ is undefined for some angles θ. Thus the identity is also undefined for these θ, and hence is not valid for these angles. Also, for some θ, we might have division by zero in the LHS, which is not allowed. Thus the identity won't hold for these angles also.

Execute the strategy

Equation:

$$\begin{aligned} LHS &= \frac{\sin\theta + 2\ \sin\theta\ \cos\theta}{1 + \cos\theta + (2\cos^2\theta - 1)} \\ &= \frac{\sin\theta(1 + 2\cos\theta)}{\cos\theta(1 + 2\cos\theta)} \\ &= \frac{\sin\theta}{\cos\theta} \\ &= \tan\theta \\ &= RHS \end{aligned}$$

We know that $\tan\theta$ is undefined when $\theta = 90^\circ + 180^\circ n$, where n is an integer. The LHS is undefined when $1 + \cos\theta + \cos 2\theta = 0$. Thus we need to solve this equation.

Equation:

$$\begin{aligned} & 1 + \cos\theta + \cos 2\theta &= 0 \\ \Rightarrow\ & \cos\theta(1 + 2\cos\theta) &= 0 \end{aligned}$$

The ab

$\theta = 9$

values

$1 + 2$

$\theta = \ldots$

identit

$\theta = \ldots$

Exercise:
Trigonometric Equations

Problem: Solve the following equation for y without using a calculator:

Equation:

$$\frac{1 - \sin y - \cos 2y}{\sin 2y - \cos y} = -1$$

Solution:

Identify a strategy

Before we are able to solve the equation, we first need to simplify the left-hand side. We do this by using the double-angle formulas.

Execute the strategy

Equation:

$$\begin{aligned} & \frac{1 - \sin y - \left(1 - 2\sin^2 y\right)}{2\sin y\cos y - \cos y} &= -1 \\ \Rightarrow\ & \frac{2\sin^2 y - \sin y}{\cos y(2\sin y - 1)} &= -1 \\ \Rightarrow\ & \frac{\sin y(2\sin y - 1)}{\cos y(2\sin y - 1)} &= -1 \\ \Rightarrow\ & \tan y &= -1 \\ \Rightarrow\ & y = 135^\circ + 180^\circ n;\ n \in \mathbb{Z} & \end{aligned}$$

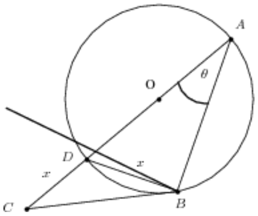
Applications of Trigonometric Functions

Problems in Two Dimensions

Exercise:
Problem in Two Dimensions

Problem: For the figure below, we are given that $BC = BD = x$.

Show that $BC^2 = 2x^2\,(1+ \sin\,\theta)$.



Solution:

Identify a strategy We want CB, and we have CD and BD. If we could get the angle $B\widehat{D}C$, then we could use the cosine rule to determine BC. This is possible, as $\triangle ABD$ is a right-angled triangle. We know this from circle geometry, that any triangle circumscribed by a circle with one side going through the origin, is right-angled. As we have two angles of $\triangle ABD$, we know $A\widehat{D}B$ and hence $B\widehat{D}C$. Using the cosine rule, we can get BC^2.

Execute the strategy **Equation:**

$$A\widehat{D}B = 180^\circ - \theta - 90^\circ = 90^\circ - \theta$$

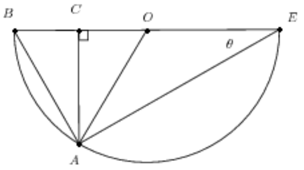
Thus **Equation:**

$$\begin{aligned} B\widehat{D}C &= 180^\circ - A\widehat{D}B \\ &= 180^\circ - (90^\circ - \theta) \\ &= 90^\circ + \theta \end{aligned}$$

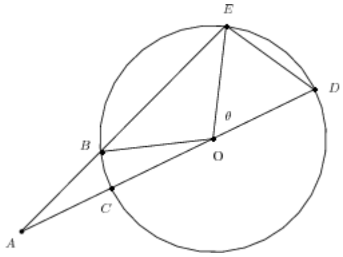
Now the cosine rule gives **Equation:**

$$\begin{aligned} BC^2 &= CD^2 + BD^2 \\ &= x^2 + x^2 - 2\,\cdot \\ &= 2x^2 + 2x^2[\,\mathrm{si} \\ &= 2x^2 + 2x^2[\,1 \\ &= 2x^2\,(1- \sin\,\theta \end{aligned}$$

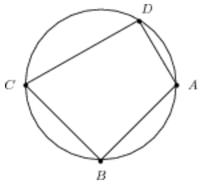
1. For the diagram on the right,
   1. Find $A\widehat{O}C$ in terms of θ.
   2. Find an expression for:
      1. $\cos\,\theta$
      2. $\sin\,\theta$
      3. $\sin\,2\theta$
   3. Using the above, show that $\sin\,2\theta = 2\,\sin\,\theta\,\cos\,\theta$.
   4. Now do the same for $\cos\,2\theta$ and $\tan\,\theta$.



2. DC is a diameter of circle O with radius r. $CA = r$, $AB = DE$ and $D\widehat{O}E = \theta$. Show that $\cos\theta = \frac{1}{4}$.



3. The figure below shows a cyclic quadrilateral with $\frac{BC}{CD} = \frac{AD}{AB}$.
    1. Show that the area of the cyclic quadrilateral is $DC \cdot DA \cdot \sin\widehat{D}$.
    2. Find expressions for $\cos\widehat{D}$ and $\cos\widehat{B}$ in terms of the quadrilateral sides.
    3. Show that $2CA^2 = CD^2 + DA^2 + AB^2 + BC^2$.
    4. Suppose that $BC = 10$, $CD = 15$, $AD = 4$ and $AB = 6$. Find CA^2.
    5. Find the angle $\widehat{D}$ using your expression for $\cos\widehat{D}$. Hence find the area of $ABCD$.



Problems in 3 dimensions

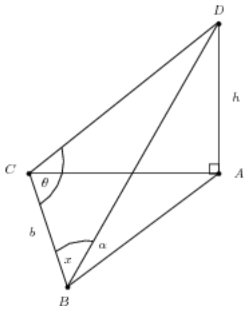
Exercise:
Height of tower

Problem:

D is the top of a tower of height h. Its base is at C. The triangle ABC lies on the ground (a horizontal plane). If we have that $BC = b$, $D\widehat{B}A = \alpha$, $D\widehat{B}C = x$ and $D\widehat{C}B = \theta$, show that

Equation:

$$h = \frac{b \sin\alpha \sin x}{\sin(x + \theta)}$$



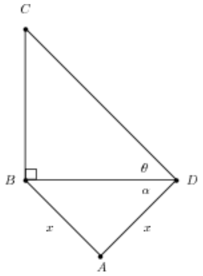
Solution:

Identify a strategy We have that the triangle ABD is right-angled. Thus we can relate the height h with the angle α and either the length BA or BD (using sines or cosines). But we have two angles and a length for

$\triangle BCD$, and thus can work out all the remaining lengths and angles of this triangle. We can thus work out BD.

Execute the strategy	We have that	**Equation:** $$\begin{aligned} \frac{h}{BD} &= \sin \alpha \\ \Rightarrow \quad h &= BD \sin \alpha \end{aligned}$$	Now we need BD in terms of the given angles and length b. Considering the triangle BCD, we see that we can use the sine rule.	**Equation:** $$\begin{aligned} \frac{\sin \theta}{BD} &= \frac{\sin \left(B\widehat{D}C\right)}{b} \\ \Rightarrow \quad BD &= \frac{b \sin \theta}{\sin \left(b\widehat{D}C\right)} \end{aligned}$$	But $D\widehat{B}C = 180^\circ - \alpha - \theta$, and	**Equati** $\sin (1$

1. The line BC represents a tall tower, with C at its foot. Its angle of elevation from D is θ. We are also given that $BA = AD = x$.



   1. Find the height of the tower BC in terms of x, $\tan \theta$ and $\cos 2\alpha$.
   2. Find BC if we are given that $x = 140m$, $\alpha = 21^\circ$ and $\theta = 9^\circ$.

Other Geometries

Taxicab Geometry

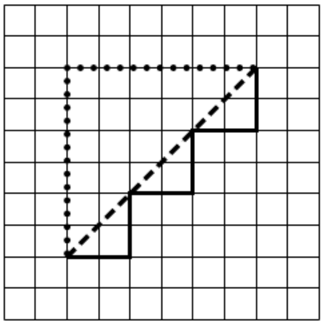
Taxicab geometry, considered by Hermann Minkowski in the 19th century, is a form of geometry in which the usual metric of Euclidean geometry is replaced by a new metric in which the distance between two points is the sum of the (absolute) differences of their coordinates.

Manhattan distance

The metric in taxi-cab geometry, is known as the *Manhattan distance*, between two points in an Euclidean space with fixed Cartesian coordinate system as the sum of the lengths of the projections of the line segment between the points onto the coordinate axes.

For example, the Manhattan distance between the point P_1 with coordinates (x_1, y_1) and the point P_2 at (x_2, y_2) is
Equation:

$$|x_1 - x_2| + |y_1 - y_2|$$



Manhattan Distance (dotted and solid) compared to Euclidean Distance (dashed). In each case the Manhattan distance is 12 units, while the Euclidean distance is $\sqrt{36}$

The Manhattan distance changes if the coordinate system is rotated, but does not depend on the translation of the coordinate system or its reflection with respect to a coordinate axis.

Manhattan distance is also known as city block distance or taxi-cab distance. It is given these names because it is the shortest distance a car would drive in a city laid out in square blocks.

Taxicab geometry satisfies all of Euclid's axioms except for the side-angle-side axiom, as one can generate two triangles with two sides and the angle between them the same and have them not be congruent. In particular, the parallel postulate holds.

A circle in taxicab geometry consists of those points that are a fixed Manhattan distance from the center. These circles are squares whose sides make a 45° angle with the coordinate axes.

The great-circle distance is the shortest distance between any two points on the surface of a sphere measured along a path on the surface of the sphere (as opposed to going through the sphere's interior). Because spherical geometry is rather different from ordinary Euclidean geometry, the equations for distance take on a different form. The distance between two points in Euclidean space is the length of a straight line from one point to the other. On the sphere, however, there are no straight lines. In non-Euclidean geometry, straight lines are replaced with geodesics. Geodesics on the sphere are the great circles (circles on the sphere whose centers are coincident with the center of the sphere). The shape of the Earth more closely resembles a flattened spheroid with extreme values for the radius of curvature, or arcradius, of 6335.437 km at the equator (vertically) and 6399.592 km at the poles, and having an average great-circle radius of 6372.795 km.

Summary of the Trigonomertic Rules and Identities

Pythagorean Identity	Cofunction Identities	Ratio Identities

$\cos^2\theta + \sin^2\theta = 1$	$\sin(90^\circ - \theta) = \cos\theta$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$
	$\cos(90^\circ - \theta) = \sin\theta$	
Odd/Even Identities	Periodicity Identities	Double angle Identities
$\sin(-\theta) = -\sin\theta$	$\sin(\theta \pm 360^\circ) = \sin\theta$	$\sin(2\theta) = 2\sin\theta\cos\theta$
$\cos(-\theta) = \cos\theta$	$\cos(\theta \pm 360^\circ) = \cos\theta$	$\cos(2\theta) = \cos^2\theta - \sin^2\theta$
$\tan(-\theta) = -\tan\theta$	$\tan(\theta \pm 180^\circ) = \tan\theta$	$\cos(2\theta) = 2\cos^2\theta - 1$
		$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$
Addition/Subtraction Identities	Area Rule	Cosine rule
$\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$	$\text{Area} = \frac{1}{2}bc\sin A$	$a^2 = b^2 + c^2 - 2bc\cos A$
$\sin(\theta - \varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi$	$\text{Area} = \frac{1}{2}ab\sin C$	$b^2 = a^2 + c^2 - 2ac\cos B$
$\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$	$Area = \frac{1}{2}ac\sin B$	$c^2 = a^2 + b^2 - 2ab\cos C$
$\cos(\theta - \varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi$		
$\tan(\theta + \varphi) = \frac{\tan\varphi + \tan\theta}{1 - \tan\theta\tan\varphi}$		
$\tan(\theta - \varphi) = \frac{\tan\varphi - \tan\theta}{1 + \tan\theta\tan\varphi}$		
Sine Rule		
$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$		

End of Chapter Exercises

Do the following without using a calculator.

1. Suppose $\cos\theta = 0.7$. Find $\cos 2\theta$ and $\cos 4\theta$.
2. If $\sin\theta = \frac{4}{7}$, again find $\cos 2\theta$ and $\cos 4\theta$.
3. Work out the following:

    1. $\cos 15^\circ$

2. $\cos 75^\circ$
3. $\tan 105^\circ$
4. $\cos 15^\circ$
5. $\cos 3^\circ \cos 42^\circ - \sin 3^\circ \sin 42^\circ$
6. $1 - 2\sin^2(22.5^\circ)$

4. Solve the following equations:

    1. $\cos 3\theta \cdot \cos \theta - \sin 3\theta \cdot \sin \theta = -\frac{1}{2}$
    2. $3\sin \theta = 2\cos^2 \theta$

5. Prove the following identities

    1. $\sin^3 \theta = \frac{3\sin\theta - \sin 3\theta}{4}$
    2. $\cos^2 \alpha \left(1 - \tan^2 \alpha\right) = \cos 2\alpha$
    3. $4\sin \theta \cdot \cos \theta \cdot \cos 2\theta = \sin 4\theta$
    4. $4\cos^3 x - 3\cos x = \cos 3x$
    5. $\tan y = \frac{\sin 2y}{\cos 2y + 1}$

6. (Challenge question!) If $a + b + c = 180^\circ$, prove that
   Equation:

$$\sin^3 a + \sin^3 b + \sin^3 c = 3\cos(a/2)\cos(b/2)\cos(c/2) + \cos(3a/2)\cos(3b/2)\cos(3c/2)$$